

IOWA STATE UNIVERSITY

Digital Repository

Graduate Theses and Dissertations

Iowa State University Capstones, Theses and
Dissertations

2010

An option-theoretic valuation model for residential mortgages with stochastic conditions and discount factors

Fernando Miranda-mendoza
Iowa State University

Follow this and additional works at: <https://lib.dr.iastate.edu/etd>



Part of the [Mathematics Commons](#)

Recommended Citation

Miranda-mendoza, Fernando, "An option-theoretic valuation model for residential mortgages with stochastic conditions and discount factors" (2010). *Graduate Theses and Dissertations*. 11328.
<https://lib.dr.iastate.edu/etd/11328>

This Dissertation is brought to you for free and open access by the Iowa State University Capstones, Theses and Dissertations at Iowa State University Digital Repository. It has been accepted for inclusion in Graduate Theses and Dissertations by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.

**An option-theoretic valuation model for residential mortgages with stochastic
conditions and discount factors**

by

Fernando Miranda-Mendoza

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

Major: Applied Mathematics

Program of Study Committee:
L. Steven Hou, Major Professor
Scott Hansen
Hailiang Liu
Sung-Yell Song
Ananda Weerasinghe

Iowa State University

Ames, Iowa

2010

Copyright © Fernando Miranda-Mendoza, 2010. All rights reserved.

TABLE OF CONTENTS

LIST OF TABLES	v
LIST OF FIGURES	vi
ABSTRACT	vii
CHAPTER 1. RELEVANT MATH FINANCE BACKGROUND	1
1.1 Overview of the Main Results	1
1.2 Basic Mortgage Terminology	4
1.3 A Brief Introduction to Options	7
1.3.1 Standard Model for Asset Prices	7
1.3.2 The Black-Scholes Equation	9
1.4 American Options	11
1.4.1 American Put Options	11
1.4.2 Free Boundary Problem	12
1.4.3 Linear Complementarity Problem	16
1.4.4 Constrained Matrix Problem	16
1.4.5 Variational Inequalities on a Hilbert Space	17
1.4.6 The American Option as a Variational Inequality	19
1.5 Interest Rate Models	20
1.5.1 Term Structure of Interest Rates	21
1.5.2 Continuous Time Interest Rate Models	23
1.5.3 Short Rate Models	25
1.5.4 The Cox-Ingersoll-Ross Model	28
1.6 Risk-Neutral Valuation	28

CHAPTER 2. STANDARD OPTION-THEORETIC MORTGAGE VALU-	
ATION	33
2.1 Mortgage Valuation Models	33
2.1.1 Previous Literature	33
2.1.2 Option-Theoretic Models	34
2.2 The Standard Option-Theoretic Mortgage Model	35
2.2.1 State Variables	36
2.2.2 Definitions, Temporal, and Boundary Conditions	37
2.2.3 Time Reversal	42
2.3 Valuation PDE	43
2.3.1 Derivation of the Valuation PDE	44
2.3.2 The Special Case of A	47
2.4 The Relevance of the Interest Rate Model in Mortgage Valuation	48
CHAPTER 3. MORTGAGE VALUATION WITH DISCOUNT FACTORS,	
COHERENT BOUNDARY CONDITIONS, AND STOCHASTIC TERMS	50
3.1 Discount Factors	50
3.1.1 Discounted Value of the Mortgage	51
3.1.2 Conditions of the Economy: $\lambda_{fin}(t)$	51
3.1.3 Liquidity of the Mortgage Market: $\lambda_{liq}(t)$	55
3.1.4 Quality of the Mortgage: $\lambda_{cr}(t)$	56
3.2 Coherent Boundary Conditions with Stochastic Terms	57
CHAPTER 4. REVIEW ON THE WIENER-ITO CHAOS EXPANSIONS	60
4.1 The White Noise Probability Space	60
4.2 Wiener Ito Chaos Expansion	62
4.3 Stochastic Distributions	64
CHAPTER 5. SIMULATION OF THE MODEL	66
5.1 Chaos Expansions Applied to the Model	66
5.1.1 Sequence of Deterministic PDEs	67

5.2	Discretization of the Coefficient PDEs	69
5.2.1	Discretization of the Domain	69
5.2.2	Numerical Scheme	71
5.2.3	Linear Complementarity Formulation	80
5.3	PSOR Method	81
5.3.1	Jacobi Iterations	82
5.3.2	Gauss-Seidel Iterations	83
5.3.3	SOR Iterations	83
5.3.4	PSOR Iterations Applied to the Linear Complementarity Problem . . .	85
5.4	Numerical Experiments	86
5.4.1	Numerical Results when r is Piecewise Constant	87
5.4.2	Numerical Results for the Full Model	93
CHAPTER 6.	CONCLUDING REMARKS	96
6.1	Mortgage Backed Securities	96
6.2	Valuation of Pass-through MBS	97
APPENDIX A.	MONTHLY PAYMENT AND PRINCIPAL BALANCE . .	99
APPENDIX B.	PSEUDOCODES	100
BIBLIOGRAPHY	102
ACKNOWLEDGEMENTS	105

LIST OF TABLES

Table 1.1	Some short rate models from the literature	26
Table 1.2	Characteristics of short rate models	27
Table 5.1	Parameters used in the simplified model	88
Table 5.2	Parameters used in the full model	93

LIST OF FIGURES

Figure 5.1	Discretization of the domain	70
Figure 5.2	Typical Crank-Nicolson stencil	72
Figure 5.3	Numbering of the unknown approximations $F_{i,j}^n$	75
Figure 5.4	Numerical results for month $k = 180$	89
Figure 5.5	Numerical results for month $k = 360$	90
Figure 5.6	The effect of k on V_0	91
Figure 5.7	The effect of c on V_0 , when $k = 180$	92
Figure 5.8	The effect of L on V_0 , when $k = 180$ months	92
Figure 5.9	The effect of L on V_0 , when $k = 180$ and $h = 96.85$	94
Figure 5.10	The effect of L on V_0 , when $k = 180$ and $\tau = 29.94$	94

ABSTRACT

Standard mathematical mortgage valuation models consist of three components: the future promised payments, the financial option to default, and the financial option to prepay. In this thesis we propose and analyze new concepts introduced into the standard models. The new concepts include discount factors, coherent boundary conditions, and stochastic terms. In this framework, the value of a mortgage satisfies a Black-Scholes type stochastic PDE. The approximate solution to our model involves a numerical method based on the Wiener-Ito chaos expansion, which breaks the stochastic PDE into a sequence of deterministic PDEs. These PDEs involve a free boundary, are discretized by finite differences, and solved through the PSOR method. Finally, extensions to MBS valuation are discussed. This work represents a timely study of mortgage valuation in the wake of the recent MBS/financial crisis.

This thesis is broadly organized as follows: In chapter 1, we briefly introduce some concepts that are part of the foundations of the standard mortgage models. In chapter 2, we review the standard mortgage valuation PDE models. In chapter 3, we discuss the discount factors, the coherent boundary conditions, and the stochastic terms. In chapter 4 we give a quick overview of the Wiener-Ito chaos expansion. In chapter 5 we analyze the simulation of our model and present some numerical results. Finally, in chapter 6 we make some remarks regarding the valuation of MBS.

CHAPTER 1. RELEVANT MATH FINANCE BACKGROUND

In this chapter we briefly introduce some concepts that are part of the foundations of the standard mortgage models.

1.1 Overview of the Main Results

In this thesis we propose, study, and analyze a PDE mortgage valuation model. Like the standard valuation models, see for instance [17], the mortgage value is broken into three components, namely the future promised payments, the financial option to default, and the financial option to prepay. This mortgage value V is dependent on time t , the property price h , and the prevalent interest rate r . In this context V satisfies the PDE (see section 2.3):

$$\begin{aligned} \frac{1}{2}h^2\sigma_h^2\frac{\partial^2 V}{\partial h^2} + \rho h\sqrt{r}\sigma_h\sigma_r\frac{\partial^2 V}{\partial h\partial r} + \frac{1}{2}r\sigma_r^2\frac{\partial^2 V}{\partial r^2} + \kappa(\theta - r)\frac{\partial V}{\partial r} \\ + (r - \delta)h\frac{\partial V}{\partial h} + \frac{\partial V}{\partial t} - rV = 0. \end{aligned} \quad (1.1)$$

This valuation PDE is defined on the time intervals $T_{i-1} < t \leq T_i$, where T_i represents the time when the $i - th$ monthly payment is due, and on the region given by $0 \leq h < \infty$ and $0 \leq r < \infty$.

The boundary conditions specified for this PDE in our model are consistent from both, the mathematical and the financial sense. Furthermore, they involve stochastic terms to better account for random fluctuations of the mortgage value at some of the boundary values. These conditions are specified by

- At $h = 0$, $V = Y(\omega)$.
- At $h \rightarrow \infty$, $V = \Phi(r, t)$, where $\Phi \rightarrow 0$ as $r \rightarrow \infty$.

- At $r = 0$, $V = \min\{h, TD(t)\}$.
- At $r \rightarrow \infty$, $V = 0$.

The terms involved in these expressions are explained in detail in chapters 2 and 3, notice however that $Y(\omega)$ is a random function.

The mortgage value also must satisfy a condition that gives rise to a free boundary. This is called the prepayment condition, as it specifies for which combination of variables prepayment must occur. This condition states that the mortgage value V must be at most the total debt TD (see subsection 2.2.2):

$$V(h, r, t) \leq TD(t), \quad \forall t. \quad (1.2)$$

Prepayment first occurs when

$$V(h, r, t) = TD(t).$$

For each fixed time, this last condition specifies a curve that divides the hr -plane into two regions, in each of these the mortgage value satisfies two properties. In one region V satisfies the PDE (1.1), while in the other region $V = TD$.

The PDE (1.1) together with the boundary conditions and the stochastic term $Y(\omega)$ imply that the mortgage value V is also dependent on the random element ω , so $V = V(h, r, t, \omega)$, and hence (1.1) is in fact a stochastic PDE. The numerical simulation of our model uses the Wiener-Ito chaos expansion to break the stochastic PDE into a sequence of deterministic PDEs. This is briefly described as follows: the mortgage value V is expanded as a series

$$V(h, r, t, \omega) = \sum_{\alpha} V_{\alpha}(h, r, t) P_{\alpha}(\omega), \quad (1.3)$$

plugging this series into the valuation PDE (1.1) results in the a sequence of deterministic PDEs for the coefficients $V_{\alpha}(h, r, t)$, where each of these coefficient functions also satisfies (1.1) with similar boundary conditions. This sequence of PDEs is described in detail in 5.1.1.

Even though we can theoretically solve the PDE (1.1) by finding all of the involved coefficients $V_{\alpha}(h, r, t)$ of the chaos expansion (1.3), in practice this is unfeasible and we will be satisfied with much less. The computation of some statistical moments, for example the ex-

pected value or the variance of V , will be sufficient for us. In fact, it must be noted that the first term of the series (1.3) is equal to $E[V(h, r, t, \omega)]$, see 4.2 and 5.1.1 for more details.

To handle the free boundary condition described in (1.2), we use the Crank-Nicolson finite differences approximation to discretize the PDE and recast this problem in its linear complementarity formulation, a visualization that permits to indirectly handle the free boundary. This linear complementarity formulation can be briefly described as

$$\begin{aligned} (TD - V) \cdot \mathcal{L}\{V\} &= 0, \\ \mathcal{L}\{V\} &\geq 0, \\ (TD - V) &\geq 0, \end{aligned} \tag{1.4}$$

where $\mathcal{L}\{V\}$ represents the PDE operator defined by the right hand side of (1.1). This allows us to find a numerical approximation to V_0 without the need of directly computing the free boundary. The PSOR method is used to find the solution to the constrained matrix problem that results from the discretizations. See 1.4.3, 1.4.4, and 5.1.1 for more details

Our model also involves discount factors $0 \leq \lambda(h, r, t; i) \leq 1$, which decrease the value of the mortgage: $V \cdot \lambda$. These discounts factors are decomposed into three components (see section 3.1):

$$\lambda = \lambda_{cr} \cdot \lambda_{liq} \cdot \lambda_{fin}, \tag{1.5}$$

each of these tied to specific aspects of the economic environment: λ_{fin} is tied to the conditions of the economy, λ_{liq} is tied to the liquidity of the mortgage market, and λ_{cr} is tied the quality of the mortgage. These factors are explained in detail in chapter 3.

The extension of this model to the valuation of mortgage backed securities (MBS) is natural, as the cash flows coming out of a mortgage pool can be directly found with our model. This is specially true for those MBS with rather similar underlying mortgages and even for those with variation on the quality of individual mortgages. See chapter 6 for a more detailed discussion.

We now present some background and terminology that will be used throughout this thesis.

1.2 Basic Mortgage Terminology

A mortgage is a financial contract by which a real estate asset is pledged as a collateral for the repayment of a loan, and this pledge is cancelled when the debt is paid in full. In the United States, this real estate asset is typically a family home, although it can also be a commercial building or a farm property. In some instances, a mortgage may be even tied to some other not necessarily residential property, for instance a ship. The mortgage contract represents a legal agreement between the lender (also called the *mortgagee*) and the borrower (also called the *mortgagor*) that specifies the repayment of the loan through a series of monthly payments, paid until a fixed date in the future (the *maturity*), and gives the right to the lender to take over the real state asset (the right of *foreclosure*) if the borrower fails to make the promised payments (if he *defaults*).

Mortgage contracts typically carry lower interest rate than other loans. This is due to the fact that the real state asset offers some security to the lender, since it acts as a *collateral*, that is to say, it could be sold in the event of default and the profit of this sale can be used to repay at least some percentage of the unpaid loan.

The borrower also has the right to terminate the mortgage contract before the specified maturity by means of *prepayment*. This means that is he can pay the remaining debt (the *total debt*) at any time before maturity. This may happen when a borrower sells the real state asset, and the buyer takes over the asset with a new mortgage contract, with perhaps a different interest rate, while the original borrower can “walk away.” More frequently, however, the lender obtains another loan that carries a lower interest rate and “discharges” the original loan, this is known as *refinancing*.

There are many types of mortgage contracts. They not only differ on the type of the real state asset that is used as a collateral, but could also have different repayment style, different type of interest rates, or different maturity. Moreover, for a fixed real estate asset, there may be some other contract specifications that are different, for example, some mortgages carry instance insurance against default or have a prepayment penalty. The most common mortgage in the U.S. consists of a loan with a constant monthly repayment and fixed interest rate, usually

called *fixed-rate mortgage* (FRM)¹

A fixed-rate mortgage consists of a loan based on a residential property, typically a house, where the contract interest rate (also called the *coupon*) is fixed and each month the lender repays a fixed amount, that is to say, the *monthly payment* is constant. This monthly payment frequently consists of three components: the interest rate payment (tied to the coupon); the scheduled repayment of the remaining fraction of the loan (this fraction is known as the *principal balance*); and an additional *servicing fee* (to pay the services of a third party that collects the monthly payment). See (A.1) in appendix A for the mathematical derivation of the monthly payment. This thesis studies FRMs only.

Two other common types of mortgages in the U.S. are the *graduated payment mortgage* (GPM) and the *adjustable-rate mortgage* (ARM). A graduated payment mortgage consists of a fixed interest rate mortgage, where the monthly payment is smaller in the initial years and it is larger for the remaining years. An adjustable-rate mortgage consists of a loan where the interest rate is periodically adjusted to reflect the changes of the prevalent interest rates in the economy. To some borrowers there is a great advantage of having GPM rather than a FRM, especially if they expect their income to increase in the future. The advantage of an ARM to some borrowers, is the prospect of lower interest payments in the event that the prevalent market interest rates decrease.

In the U.S. the typical length or *term* of a mortgage loan is 30 years, although longer and shorter term mortgages are also offered, for instance 15 year long loans are also common. The mortgage loan *amortizes* over its term, that is to say it gradually decreases by each monthly payment until the debt is completely paid at maturity.

In addition to the lender and the borrower, there may be some other players in a mortgage loan. The lender is typically a bank, or other lending institution. The borrower is typically an individual homebuyer or a family. A bank may lend using its own resources, but in the U.S., especially before the financial crisis of 2008, it may sell the loan to another party interested in receiving the stream of cash payments from the borrower. This additional player is typically

¹Throughout this thesis we will be using abbreviations such as this one when there is no risk of confusion.

an investor, who receives the mortgage payments as a *security*. Securitization is a process that distributes the risk by aggregating several mortgage loans in a pool, the investor can then buy a share on this pool. These shares or securities are called *mortgage-backed securities* (MBS). In the U.S., the largest firms that securitize loans are the two government sponsored enterprises firms Federal National Mortgage Association (FNMA) also called *Fannie Mae* and the Federal Home Loan Mortgage Corporation (FHLMC), also known as *Freddie Mac*. Also, the government owned firm, the Government National Mortgage Association (GNMA), or *Ginnie Mae* securitizes loans, where the securities are backed by the *full faith and credit* of the U.S. government (that is to say, they are insured against default). Besides the investors, there are also mortgage insurers, who will typically protect investors against the possible default of a borrower or a group of borrowers in a pool of mortgages.

A FRM loan typically does not provide the full value of home, a borrower must cover the difference between the loaned amount and the value of the home, this is known as a *down payment*. The *loan-to-value* (LTV) ratio is the loan divided by the value of the home, and it is an important quantity that lenders consider before approving a mortgage loan, as it assesses the risk of mortgages.

In mortgage valuation, the *value of a mortgage* is the value as seen from the perspective that a lender or an investor have. This value is not simply the sum value of due monthly payments, since a borrower may terminate the contract early by either prepayment or default.

This thesis develops a model to find this unknown value. This model visualizes the mortgage value as consisting of financial *options*. An option is one of the most important concepts in mathematical finance. In the next section we now present a brief overview of these related concepts.

There are many more mortgage-related terminology, for more definitions and concepts see [11], [13], [16], and [25].

1.3 A Brief Introduction to Options

The simplest financial option, a *European call option*, is a contract with the following conditions: At a prescribed time in the future, the *expiration date*, the holder of an option may purchase a prescribed asset, known as the *underlying asset*, for a prescribed amount, known as the *exercise* or *strike price*. The word “may” implies that for *the holder* of the option, this contract is a right, not an obligation. The other party to the contract, *the writer*, does have a potential obligation, as he must sell the asset if the holder chooses to buy it. Since an option confers on its holder a right with no obligation it has some value, moreover, the writer of an option must be compensated for the obligation assumed. The right to sell an asset is known as a *put option*. Both, call and put options are valued by the model, where given the value of the underlying asset to we can then derive a valuation PDE, the Black-Scholes equation (see [2]). The boundary conditions of this PDE make the distinction among the type of options that are valued.

1.3.1 Standard Model for Asset Prices

The basic assumption that is common to most of the option pricing theory, is that we do not know and cannot predict tomorrow’s values of the underlying asset prices due to the complexity of the financial market. The past history of the asset price is available and can be examined, but cannot be used to forecast the next move the asset will make. However, it is possible from the examination of the past prices to predict what are the likely jumps in the asset price, what are its mean and variance and what is the likely statistical distribution of future asset prices.

Almost all option pricing models are founded on one simple model for asset price movements, involving parameters derived, for example, from historical or market data. It is assumed that the asset prices must move randomly because of the *efficient market hypothesis* which can be briefly described by the two statements:

- The past history is fully reflected in the present price, which does not hold any further information.

- Markets respond immediately to any new information about an asset.

Hence, according to this economics hypothesis, the modelling of asset prices can be interpreted as the modelling of the arrival of new information which affects the asset price. With these two assumptions, changes in the asset price define a Markov process.

For the valuation of an asset, the absolute change in the asset price is not by a useful quantity. Instead, most models associate a *return* to each change in asset price, defined to be the change in the price divided by the original value. Informally this can be described as follows: if at time t the asset price is S , and during a subsequent time interval dt the price changes to $S + dS$, the goal is to model the return defined as dS/S .

The simplest model for valuing the return dS/S decomposes it into two parts. One is the predictable, deterministic and anticipated return similar to the return on money invested in a risk-free bank. It gives a contribution μdt , to dS/S , where μ is a measure of the average rate of growth of the asset price, also known as the *drift*. The second contribution to dS/S models the random change in the asset price in response to external effects, such as unexpected news. It is represented by a random sample drawn from a normal distribution with mean zero and adds a term σdX to dS/S , where σ is the so-called *volatility*, which measures the standard deviation of the returns.

The term dX contains the randomness of the asset process is a standard Wiener process. It has the following properties: dX is normally distributed, the mean of dX is zero, and the variance of dX is dt .

Putting the contributions to the return dS/S together gives rise to the stochastic differential equation

$$\frac{dS}{S} = \mu dt + \sigma dX$$

or, as it is commonly written in the literature:

$$dS = \mu S dt + \sigma S dX. \tag{1.6}$$

A practical justification of this model of asset prices is that it fits real data very well especially for the so-called equities and indices. There are some discrepancies however, for instance,

real data appears to have a greater probability of large rises or falls than the model predicts. Nevertheless, this model has stood the test of time and is widely used in the literature, as it can be the starting point for more sophisticated models.

The equation (1.6) does not refer to the past history of the asset price; the next price $S + dS$ only depends on today's price. This independence from the past is called the *Markov property*.

The mean of dS is $E[dS] = \mu S dt$, since $E[dX] = 0$. On the average, the next value for S is higher than the old value by an amount of $\mu S dt$. The variance of dS is $\text{Var}[dS] = \sigma^2 S^2 dt$.

1.3.2 The Black-Scholes Equation

We can now derive a partial differential equation (PDE) for the pricing of options, the *Black-Scholes equation*, described first in [2]. The mortgage valuation PDE (1.1) that will be described in detail later in chapter 2, is analogous to the Black-Scholes PDE presented here.

Let $V(S, t)$ represent the price of an option at time t written on an underlying asset whose price S follows the process given by equation (1.6). This option can be either a call or put option, as both kind of option values satisfy the Black-Scholes equation, the difference appears between the two kinds lies on the boundary conditions. The Ito's lemma², implies that V satisfies the SDE:

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt + \sigma S \frac{\partial V}{\partial S} dX. \quad (1.7)$$

Next, we construct a portfolio consisting of one option and a number (to be specified below) $-\Delta$ of the underlying asset. The value of this portfolio is then:

$$\Pi = V - \Delta S. \quad (1.8)$$

The jump in the value of this portfolio in one small time-step is:

$$d\Pi = dV - \Delta dS, \quad (1.9)$$

²Ito's lemma applied to a function $f(S)$ that depends on a random process S , which is described by

$$dS = A(S, t)dt + B(S, t)dX,$$

states that

$$df = \left(A \frac{\partial f}{\partial S} + \frac{1}{2} B^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} \right) dt + B \frac{\partial f}{\partial S} dX.$$

where Δ is held fixed, as the time-step dt is assumed to be small. Combining expressions (1.6) and (1.7) together with (1.9) above, we observe that randomness is eliminated from this portfolio by choosing

$$\Delta = \frac{\partial V}{\partial S}. \quad (1.10)$$

We now assume the following *no-arbitrage or arbitrage free argument* is valid: The return on the portfolio and the return on a riskless bank account is the same. Hence:

$$r\Pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \quad (1.11)$$

Combining (1.8), (1.10), and (1.11) yields the Black-Scholes partial differential equation (see [2] and [30] for more details):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (1.12)$$

The elimination of randomness and the arbitrage assumption used above are both standard financial arguments. All the random terms can be cancelled from equation (1.9) by the very special choice of the so-called *option delta* (1.10). The "no-arbitrage argument" states that in a financial market there are no opportunities to make instantaneous risk-free profits and yields equation (1.11). We visualize an option as a *derivative asset*, while the stock is seen as the *fundamental asset*, and hence an option is just a package of possible payouts. In principle, the same pattern of payouts can be obtained through the continuous adjustment of a portfolio consisting of a riskless investment and the underlying asset. More details about these two financial ideas are given in section 1.6.

To complete the derivation of the Black-Scholes PDE (1.12), we must include final and boundary conditions. The final condition comes from the information at the expiration of the option, since we know the option final value, which depends on whether the option is exercised or not. The boundary conditions arise from both financial and mathematical considerations, see [29] for more details.

We now present the standard final and boundary conditions for a call and a put option:

- For a European call option with value $C(S, t)$, exercise price E , and expiration date T :

- Final condition: $C(S, T) = \max\{S - E, 0\}$.
- Boundary conditions: $C(0, t) = 0$ and $C(S, t) \sim S$, as $S \rightarrow \infty$.
- For a European put option with value $P(S, t)$, exercise price E , and expiration date T :
 - Final condition: $P(S, T) = \max\{E - S, 0\}$.
 - Boundary conditions: $P(0, t) = Ee^{-r(T-t)}$ and $P(S, t) \rightarrow 0$, as $S \rightarrow \infty$.

1.4 American Options

The standard mortgage models described in chapter 2 regard the borrower's right to prepay as an American type option. These kind of option is typically valued by solving the Black-Scholes PDE (1.12), but unlike the relatively simple boundary conditions that the value of a European option must satisfy, the value of an American option must satisfy a so-called free boundary condition. This results in a *free boundary problem* that is harder to solve than a typical boundary value problem, as we must in principle find both, the solution to the PDE and the unknown free boundary.

In this section we will describe how a basic American put option can be modeled as a free boundary problem. Moreover, we will show the equivalence of the resulting free boundary problem to a *linear complementarity problem (LCP)*, which provides us with a powerful numerical method where the unknown free boundary does not need to be found explicitly. We will also show the equivalence of the linear complementarity problem to a *variational inequality*, this reinterpretation of the problem guarantees the existence and uniqueness of a solution to the free boundary problem.

1.4.1 American Put Options

Recall that (see section 1.3) a financial contract that gives its holder the right to sell an asset for a specified price is called a put option. We now present the valuation of model of an American put option. An American option is one that allows its holder to exercise it at any time before expiry. This additional early exercise feature of an American option gives its

holder greater rights than those of a European option, so an American option potentially has a higher value than a European option. Hence, when early exercise is permitted, one must impose the following constraint:

$$P(S, t) \geq \max\{S - E, 0\}, \quad (1.13)$$

where $P(S, t)$ is the value of the American put option for asset price S and at time t , and E is the strike price. Otherwise, if $P(S, t) < \max\{S - E, 0\}$ there is an arbitrage opportunity, as one can buy the asset for price S , while at the same time buy the option for price P , and then immediately exercise the option by selling the asset for E , thereby making a risk free profit of $E - P - S$. Hence, the free boundary condition (1.13) must indeed hold for all S and t .

As mentioned before, the valuation of American options is rather complicated, since at each time t we must determine not only the value of the option P , but also, for each value of S , whether or not it should be exercised. Typically at each time t there is a particular value of S which marks the boundary between the following two regions: to one side one should hold the option and to the other one should exercise it. We will denote this value by $S^* = S^*(t)$, the so-called *optimal exercise price*.

1.4.2 Free Boundary Problem

The problem of valuing an American option can be uniquely determined by the following list of constraints:

- The option value must be greater than or equal to the payoff function (see (1.13)).
- The Black-Scholes equation (equation (1.12)) must be replaced by an inequality.
- The option value $P(S, t)$ must be a continuous function of S for all fixed time t .
- The option delta $\frac{\partial P}{\partial S}$ (see (1.10)) must be continuous for all S and t .

All these constraints are a consequence of financial reasoning and no-arbitrage considerations. We already briefly explained the first constraint, this is precisely inequality (1.13). For more details on the rest of the constraint see [30].

These constraints specify a problem whose solution is the value of an American put option, this is the following free boundary problem:

- For $0 \leq t < T$ and $0 \leq S < S^*(t)$,

$$P = E - S, \quad \text{early exercise is optimal,} \quad (1.14)$$

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP < 0. \quad (1.15)$$

- For $0 \leq t < T$ and $S^*(t) < S < \infty$,

$$P > E - S, \quad \text{early exercise is not optimal,} \quad (1.16)$$

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP = 0. \quad (1.17)$$

- Boundary conditions at $S^*(t)$:

$$P(S^*(t), t) = \max\{E - S_f(t), 0\}, \quad (1.18)$$

$$\frac{\partial P}{\partial S}(S^*(t), t) = -1. \quad (1.19)$$

We can think of these two as being one boundary condition to determine the option value on the free boundary, and another condition to determine the location of the free boundary, which is given by the set of points of the form $(S^*(t), t)$.

- Boundary condition as $S \rightarrow \infty$:

$$\lim_{S \rightarrow \infty} P(S, t) = 0. \quad (1.20)$$

- Terminal condition at $t = T$:

$$P(S, T) = \max\{E - S, 0\}. \quad (1.21)$$

Condition (1.18) that $\frac{\partial P}{\partial S}(S^*(t), t) = -1$ is not implied by the fact that $P(S^*(t), t) = E - S^*(t)$, as one does not know *a priori* the exact value of $S^*(t)$, and so an additional condition is needed to determine it. No-arbitrage arguments (see [30]) show that the gradient of P must be continuous, and this gives rise to this additional condition.

Notice that when $S < S^*(t)$ we completely know the value of the American put option, and this value P is given by (1.14). From our perspective, in principle we do not need to be concerned with inequality (1.15). On the other hand, when $S^*(t) < S$ we only know that P must satisfy both, inequality (1.16) and the Black-Scholes PDE (1.17) and in principle we must solve this PDE. Of course, as pointed out before, we also need to either explicitly find $S^*(t)$ for all t or find a way to handle it indirectly.

We now follow the approach of [20] and [21] and summarize conditions (1.14) through (1.21) as follows:

- The complete value of the American put option is given by

$$P_c(S, t) = \begin{cases} \max\{E - S, 0\}, & \text{if } S \in [0, S^*(t)), \\ P(S, t), & \text{if } S \in [S^*(t), \infty), \end{cases} \quad (1.22)$$

- $S^*(t)$ is the unknown free boundary and $P(S, t)$ is determined by the Black-Scholes equation:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP = 0, \quad t \in [0, T].$$

- The terminal and boundary conditions are given by:

$$P(S, T) = \max\{E - S, 0\},$$

$$S^*(T) = E,$$

$$P(S^*(t), t) = E - S^*(t),$$

$$P_S(S^*(t), t) = -1,$$

$$\lim_{S \rightarrow \infty} P(S, t) = 0.$$

The complete value of the option $P_c(S, t)$, given by expression (1.22), is indeed the solution to the free boundary problem described by conditions (1.14) through (1.21).

We will use numerical analysis techniques to find an numerical approximation to $P_c(S, t)$. To simplify the analysis and manipulations that will follow, we will transform the original variables (S, t) to the dimensionless variables (x, τ) . This is a standard procedure in the literature, see for instance [29] and [30].

The change of variables is given by the following formulas: $S = Ee^x$, $t = T - \frac{2\tau}{\sigma^2}$, and $P(S, t) = Ev(x, \tau)$. This gives rise to an intermediate PDE

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv,$$

where $k = \frac{2r}{\sigma^2}$, and the terminal condition becomes an initial condition $v(x, 0) = \max\{e^x - 1, 0\}$. We perform an additional transformation given by $v = e^{\alpha x + \beta \tau} u(x, \tau)$, where $\alpha = -\frac{1}{2}(k-1)$ and $\beta = -\frac{1}{4}(k+1)^2$. With this new change, the payoff function $\max\{E - S, 0\}$ becomes

$$g(x, \tau) = e^{\frac{1}{2}(k+1)^2\tau} \max\left\{e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0\right\}.$$

These changes yield the following PDE:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad (1.23)$$

with initial condition given by

$$u(x, 0) = g(x, 0) = \max\left\{e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0\right\}, \quad (1.24)$$

while the free boundary $S = S^*(t)$ becomes $x = x^*(\tau)$ and is given by the relation $x^*(t) = \ln\left(\frac{S^*(t)}{E}\right)$, moreover $x^*(0) = 0$.

By the above transformations, and from equations (1.23) and (1.24) the value of an American option can be found by the function:

$$u_c(x, \tau) = \begin{cases} g(x, \tau), & \text{if } x \in (-\infty, x_f(t)), \\ u(x, \tau), & \text{if } x \in [x_f(t), \infty), \end{cases} \quad (1.25)$$

where $u(x, \tau)$ is the solution to PDE (1.23), with the following initial and boundary conditions:

$$\begin{aligned} u(x, 0) &= g(x, 0), \\ \lim_{x \rightarrow \infty} u(x, \tau) &= 0, \\ \lim_{x \rightarrow -\infty} u(x, \tau) &= \lim_{x \rightarrow -\infty} g(x, \tau), \end{aligned} \quad (1.26)$$

and, furthermore, both u and $\frac{\partial u}{\partial x}$ must be continuous on $x^*(\tau)$ for all τ . Finally, the constraint given by inequality (1.13), which must hold for all x and τ becomes:

$$u(x, \tau) \geq g(x, \tau). \quad (1.27)$$

1.4.3 Linear Complementarity Problem

Any problem of the form

$$\mathcal{A}\mathcal{B} = 0, \quad \mathcal{A} \geq 0, \quad \mathcal{B} \geq 0,$$

is called a *complementarity problem*. If at least one of \mathcal{A} and \mathcal{B} is a linear operator, the problem is called *linear complementarity problem (LCP)*.

The transformed problem given by (1.23), (1.25), and (1.27) can be rewritten as a linear complementarity problem with no explicit mention of the free boundary $x^*(\tau)$. This is accomplished by a simple observation: From constraint (1.27), we have that either $u > g$, in which case $u_\tau - u_{xx} = 0$, or $u = g$, which in turn yields $u_\tau - u_{xx} = g_\tau - g_{xx} \geq 0$ (this last inequality is derived by direct calculation, see [29], for details). We can then recast the problem in a linear complementarity version:

$$\begin{aligned} (u_\tau - u_{xx}) \cdot (u - g) &= 0, \\ u_\tau - u_{xx} &\geq 0, \\ u - g &\geq 0, \end{aligned} \tag{1.28}$$

with the same initial and boundary conditions as given by (1.26) and the same continuity requirement on u and $\frac{\partial u}{\partial x}$.

As outlined above, the transformed problem given by expressions (1.23), (1.25), and (1.27) is indeed equivalent to this LCP, so solving the latter problem gives a solution to the former. Once again, we emphasize that the advantage of the LCP formulation over the free boundary problem is the absence of any explicit mention of the free boundary $x^*(\tau)$ and therefore the numerical analysis involved is greatly simplified. This approach is somehow standard in the literature on free boundary problems and American options, see [8], [10], and [29] for more.

1.4.4 Constrained Matrix Problem

We now use the Crank-Nicolson finite difference approximation to discretize the PDE and the inequalities given by equations (1.28), we then obtain a discrete version of these equations and we can think of it as constrained matrix problem, as the discretization of the PDE gives

rise to a matrix problem (see [27]). These constrained matrix problem is given by the following expressions:

$$\begin{aligned} (\bar{u}^{n+1} - \bar{g}^{n+1}) \cdot (C\bar{u}^{n+1} - \vec{b}^n) &= 0, \\ C\bar{u}^{n+1} &\geq \vec{b}^n, \\ \bar{u}^{n+1} &\geq \bar{g}^{n+1}. \end{aligned} \tag{1.29}$$

In these expressions, \bar{u}^{n+1} and \bar{g}^{n+1} represent the $n+1$ -level discrete approximation to $u(x, \tau)$ and $g(x, \tau)$, respectively; \vec{b}^n is the vector of “right-hand sides” of the finite difference equations; C is the square, tridiagonal, symmetric matrix that arises from the finite difference scheme. These inequalities are meant to be entriwise. See [27] and [30] for more details.

To solve the problem given by (1.29), at each time step, we calculate \bar{g}^n and \vec{b}^n from the already known values \bar{u}^n , and we then solve the same problem for \bar{u}^{n+1} . We will use the *projected successive over-relaxation (PSOR)* algorithm to obtain a sequence of equations $\{\bar{u}^{n+1,k}\}_k$ that are iterated until the difference $\|\bar{u}^{n+1,k+1} - \bar{u}^{n+1,k}\|$ is negligible. We will then set $\bar{u}^{n+1} = \bar{u}^{n+1,k+1}$, this gives the solution to $C\bar{u}^{n+1} = \vec{b}^n$. We will describe the PSOR algorithm later in section 5.3, for more on this and other iterative methods see [18].

The solution obtained by the PSOR algorithm guarantees that both $\bar{u}^{n+1} \geq \bar{g}^{n+1}$ and $(\bar{u}^{n+1} - \bar{g}^{n+1}) \cdot (C\bar{u}^{n+1} - \vec{b}^n) = 0$ hold. The other condition that $C\bar{u}^{n+1} \geq \vec{b}^n$ follows as a consequence of the structure of the matrix C , as it is positive definite matrix.

The approach briefly outlined here is the same we will adopt later in 5 to find a numerical approximation to the solution of the mortgage valuation problem.

1.4.5 Variational Inequalities on a Hilbert Space

The equivalence of the free boundary problem described by (1.25), (1.23), (1.26), and (1.27) and the linear complementarity problem described by (1.28) and (1.26) is derived from a straightforward observation made in 1.4.3. The existence and uniqueness of a solution to the linear complementarity problem, however, involves deeper mathematics. We will now briefly describe the necessary mathematical background in the general setting of Hilbert spaces. For more details on this topic see [19].

We first start with a few remarks regarding bilinear forms. A function $a : H \times H \longrightarrow \mathbb{R}$, where H is a real Hilbert space, is called a bilinear form if $a(\cdot, v)$ is linear for all v and $a(u, \cdot)$ is linear for all v . It is *bounded* if there exists a constant $\beta > 0$ such that $|a(u, v)| \leq \beta \|u\| \|v\|$, for all $u, v \in H$. This bilinear form is *symmetric* if $a(u, v) = a(v, u)$, for all $u, v \in H$. Finally, it is *coercive* if there exists a constant $\alpha > 0$ such that $a(u, v) \geq \alpha \|v\|^2$, for all $v \in H$.

In general we can describe a *variational inequality problem* as follows: Find $u \in K$, such that

$$a(u, v - u) \geq \ell(v - u), \quad v \in K, \quad (1.30)$$

where a is a bounded and coercive bilinear form, ℓ is a bounded linear functional, and $K \subset H$ is closed and convex.

There is a closely related minimization problem: Find $u \in K$, such that

$$J(u) = \min_{v \in K} J(v), \quad (1.31)$$

where

$$J(v) = \frac{1}{2}a(v, v) - \ell(v). \quad (1.32)$$

This latter problem is, in fact, equivalent to the variational inequality problem given by (1.30) when the bilinear form a is, in addition to bounded and coercive, also symmetric. The summarize this as a theorem:

Theorem 1.4.1. *There exists a unique solution to the variational inequality problem described by (1.30). Moreover, if the bilinear form a is symmetric, then the problem given by (1.30) is equivalent to the problem described by (1.31) and (1.32).*

A proof of this theorem can be found in [10]. This theorem guarantees the existence and uniqueness of a solution to the LCP given by (1.28) and (1.26), and hence also guarantees the existence and uniqueness of a solution to the original American valuation problem given by conditions (1.14) through (1.21).

In addition to the existence and uniqueness that we can derive from theorem 1.4.1, we also have a *stability result* that is important from the perspective of numerical analysis, we summarize it as another theorem:

Theorem 1.4.2. *The mapping $\ell \mapsto u$ is Lipschitz, that is to say, if u_1 and u_2 are two solutions to the variational inequality (1.30) with ℓ_1 and ℓ_2 respectively, then*

$$\|u_1 - u_2\| \leq \frac{1}{\alpha} \|\ell_1 - \ell_2\|_{H^*}, \quad (1.33)$$

where H^* denotes the space of bounded linear functionals on H .

See [19] for a proof. This fact, and in particular inequality (1.33) guarantees the stability of the numerical scheme that we described in 1.4.4.

We remark that the mortgage valuation model presented later in chapters 2 and 3 also has a unique solution due to theorems 1.4.1 and 1.4.2.

1.4.6 The American Option as a Variational Inequality

Before we conclude this section, we will justify the equivalence between the linear complementarity formulation given by (1.28) and (1.26) and a variational inequality like (1.30). Since the PDE involved in (1.28) is parabolic, the literature often refers to the inequality we will derive here as a *parabolic variational inequality*. For more details see [29].

We first define a set \mathcal{K} as the space of *test functions* $\phi(x, \tau)$ which are defined by the following conditions:

- $\phi(x, \tau)$ and $\frac{\partial \phi(x, \tau)}{\partial \tau}$ are continuous.
- $\frac{\partial \phi(x, \tau)}{\partial x}$ is piecewise continuous.
- $\phi(x, \tau) \geq g(x, \tau)$, for all x and τ .
- $\lim_{x \rightarrow \infty} \phi(x, \tau) = \lim_{x \rightarrow \infty} g(x, \tau) = 0$.
- $\lim_{x \rightarrow -\infty} \phi(x, \tau) = \lim_{x \rightarrow -\infty} g(x, \tau)$.
- $\phi(x, 0) = g(x, 0)$.

We remark that any solution to the LCP (1.28) and (1.26) problem belongs to this set \mathcal{K} . Now, note that for any $\phi(x, \tau) \in \mathcal{K}$, since $\phi \geq g$,

$$(u_\tau - u_{xx}) \cdot (\phi - g) \geq 0.$$

So, for any $\tau \in [0, T]$,

$$\int_{-\infty}^{\infty} (u_{\tau} - u_{xx}) \cdot (\phi - g) dx \geq 0. \quad (1.34)$$

On the other hand, we also have that

$$\int_{-\infty}^{\infty} (u_{\tau} - u_{xx}) \cdot (u - g) dx = 0, \quad (1.35)$$

since the integrand is equal to 0 by (1.28). Subtracting (1.34) and (1.35), yields:

$$\int_{-\infty}^{\infty} (u_{\tau} - u_{xx}) \cdot (\phi - u) dx \geq 0, \quad \forall \phi \in \mathcal{K}. \quad (1.36)$$

Integrating (1.36) by parts gives

$$\int_{-\infty}^{\infty} u_{\tau}(\phi - u) + u_x(\phi_x - u_x) dx - [u_x(\phi - u)]_{-\infty}^{\infty} \geq 0,$$

but since ϕ and u both go to 0 when $x \rightarrow \pm\infty$, then $[u_x(\phi - u)]_{-\infty}^{\infty} = 0$, hence

$$\int_{-\infty}^{\infty} u_{\tau}(\phi - u) + u_x(\phi_x - u_x) dx \geq 0, \quad \forall \phi \in \mathcal{K}. \quad (1.37)$$

The parabolic variational inequality for the function $u(x, \tau)$ is precisely equation (1.37). The left hand side can be identified as a bilinear form on the space of test functions \mathcal{K} . So, the problem of valuing an American put option can indeed be transformed to a variational inequality problem. The same can be said of the mortgage valuation model of this thesis, the proof is entirely analogous to the one outline here.

1.5 Interest Rate Models

The valuation of mortgages is intimately tied to the modeling of interest rates, as a mortgage is a loan and every loan comes with an interest rate attached to it. In the case of fixed-rate mortgage (FRM) the mortgage interest rate is fixed and set at the beginning or *origination* of the contract, and this determines among other things, the monthly payment a borrower must make. Moreover, the borrower's right to prepay is heavily influenced by how the economy's interest rates change throughout the life of the contract, as it may be advantageous to refinance if the prevailing interest rates are significantly smaller than the original mortgage rate.

In this section we will discuss a few aspects of interest rate modeling and will describe the Cox-Ingersoll-Ross continuous time model, a model that will be part of the foundations of the mortgage valuation model presented in chapters 2 and 3.

1.5.1 Term Structure of Interest Rates

In a financial market, a lender will not lend money for free, as the value of money today is potentially higher than the value of money in the future, therefore he must be compensated for the loss of potential opportunities that he misses due to a borrower using his money instead of him. This compensation is usually a percentage of the lent money that reflects the current and possible future conditions of the economy. Hence, an interest rate can be described as the price a lender charges for borrowed money.

There are several interest rates, for instance banks charge their most trustworthy customers the *prime rate*, while they charge dubious customers the *sub-prime rate*. As already mentioned, a FRM comes with an interest rate called the *coupon*. In the U.S. banks charge each other for overnight transactions the *federal funds rate*, while in the United Kingdom the *London Interbank Offered Rate (LIBOR)* is used for a similar purpose. Smaller financial institutions also charge interest rates depending on their individual criterion. Bonds also carry interest rates that in general are not the same as the ones used by banks and financial institutions. Despite this seemingly large variety of interest rates, it turns out that they are all related closely related. An interest rate model is a mathematical model that attempts to describe how all the different interest rates evolve through time, in principle is an important interest rate changes dramatically, it is expected the other rates would change too. The study of the connection among all the different rates is closely tied to the so-called *term structure of interest rates*.

The term structure of interest rates concerns the relationship among the *yields* of default-free securities, typically bonds, that differ only with respect to their term to maturity. This relationship is also popularly known as the *shape of the yield curve*. Historically, three competing theories have attracted the widest attention among economists:

- *The Expectations Theory*: This theory states that shape of the yield curve can be explained by investors' expectations about future interest rates. In mathematical terms, the most popular version says that $e^{F(0,S,S+1)} = E[e^{R(S,S+1)}|\mathcal{F}_0]$, where $F(t,T,S)$ denotes the forward rate and $R(t,T)$ the yield, two concepts that we will introduce in the next subsection (see (1.38) and (1.40)), and \mathcal{F}_t represents the information up to time t .
- *The Liquidity-Preference Theory*: This theory claims that short-term securities are more desirable to investors than longer-term securities because the former are more *liquid* (i.e. they are easier to sell and buy). In other words, investors usually prefer short-term investments, as they do not like to tie their money for too long. This implies that the prices of longer-term bonds tend to be more volatile than the prices of short-term bonds, as investors will only invest in more volatile securities if they have a higher expected return, often referred to as the *risk premium*, to offset the higher risk.
- *The Market Segmentation Theory*: Also called *The Hedging-Pressure or Preferred Habitat Theory*, it postulates that there is no reason for term premiums to be necessarily positive or to be increasing functions of maturity. In other words, each investor has in mind an appropriate set of bonds and maturity dates that are suitable for his purpose. Different groups of investors can act in different ways, but there is no reason why there should be any interaction between different groups. This means that prices of bonds with different maturity will change in unrelated ways.

These economic theories can be put together into a mathematically precise theory:

- *Arbitrage-Free Pricing Theory*: It states that the shape of the yield curve can be derived from the pricing of bonds in a market that is free of arbitrage. Hence, the future behavior of interest rates can be derived from the the prices of bonds.

This latter theory is adopted by most of the mathematical finance literature dealing with mortgage valuation and it is the approach we will follow in this thesis. See [5] and [22] for more details on the term structure of interest rates.

1.5.2 Continuous Time Interest Rate Models

We will now give the mathematical definitions of some of the standard interest rate models found in the literature. All the models presented here are continuous time models, as we focus on valuation PDEs and therefore discrete interest rate models are beyond the scope of this thesis.

As already mentioned in the previous section, the expectations theory implies that the cost of borrowing money depends on the time to maturity of the interest rate contracts available. In general it also depends on random fluctuations of the financial markets. The most basic interest rate contract can be described as a contract where a borrower pays now and then receives a “large sum” at a later fixed date, the most common example of this is a *zero-coupon bond* with maturity at T (also called a T -bond), which is a contract that guarantees its holder to be paid a fixed amount (in the literature this is assumed to be \$1 for simplicity) at time T . In mathematical terms, if we let $P(t, T)$ denote the price of a zero-coupon bond at time t , where $0 \leq t \leq T$. In general, this is a two parameter stochastic process. Its *face value* is \$1 and $P(T, T) = 1$.

The simplest interest mode assumes that the interest rate of a zero coupon bond is a constant $r > 0$. In this case we have that $P(t, T) = e^{-r(T-t)}$, and hence the constant interest rate can be expressed as $r = -\frac{\log P(t, T)}{T-t}$. In reality however, bond interest rates are not constant, in which case the interest rate is called the *yield* or *spot rate*, and it is given by the formula:

$$R(t, T) = -\frac{\log P(t, T)}{T - t}. \quad (1.38)$$

In financial terms, $R(t, T)$ is a continuously compounded interest rate.

Now, consider a bond with price $P(t, t + \Delta t)$, then from the yield (1.38) we have that

$$R(t, t + \Delta t) = -\frac{\log P(t, t + \Delta t)}{\Delta t},$$

letting $\Delta t \rightarrow 0$ and using the fact that $P(t, t) = 1$ we then have

$$r(t) = R(t, t) = -\frac{\partial}{\partial T} \log P(t, T) \Big|_{T=t}. \quad (1.39)$$

$r(t)$ is known as the *short rate* or the *instantaneous rate*. We can think of this as the daily interest rate offered by a bank to general customers.

Next, the *forward rate* at time t which applies between times T and S ($t \leq T < S$) is given by:

$$F(t, T, S) = \frac{1}{S - T} \log \frac{P(t, T)}{P(t, S)}, \quad (1.40)$$

and the *instantaneous forward rate*, in turn, is expressed as:

$$f(t, T) = -\frac{\partial}{\partial T} \log P(t, T). \quad (1.41)$$

The forward rate arises when dealing with *forward contracts*, where an investor agrees to pay \$1 at time T in return for $e^{(S-T)F(t,T,S)}$ at time S , that is to say, the interest rate between the times T and S is fixed in advance at time t . The instantaneous forward rate can be thought of as the interest rate of a contract made at time t to earn with a rate $f(t, T)$ per time unit between times T and $T + dt$. Of course, this is a theoretical concept, but it is introduced in the literature for the convenience of bond pricing, as it is easier to use $f(t, T)$ rather than $F(t, T, S)$.

Notice that, from (1.38) we have that $-\log P(t, T) = (T - t)R(t, T)$, so combining this with (1.41) gives the following alternate expression for the instantaneous forward rate:

$$f(t, T) = R(t, T) + (T - t) \frac{\partial R(t, T)}{\partial T}. \quad (1.42)$$

Integrating both sides of (1.42) with respect to t , observe that the yield and the instantaneous forward rate are related by

$$R(t, T) = -\int_t^T f(t, u) du. \quad (1.43)$$

Finally, from (1.38) and (1.41) we can derive the following two bond pricing formulas involving these two rates:

$$\begin{aligned} P(t, T) &= e^{-R(t, T)(T-t)}, \\ P(t, T) &= e^{-\int_t^T f(t, u) du}. \end{aligned}$$

In the standard interest rate models, there is a one-to-one relationship between yields $R(t, T)$ and bond prices $P(t, T)$. So, if we are able to accurately model the future behavior of bond prices, then by (1.38) we will be able to model the yield.

In the literature, however, many of the most popular interest rate models focus only on the short rate $r(t)$. Notice that the short rate $r(t)$ does not have a one-to-one correspondence to $P(t, T)$, nevertheless it can be proved that this rate is good enough for bond pricing modeling. A short rate model is given by specifying a *stochastic differential equation (SDE)* for the short rate $r(t)$, which is derived from the zero-coupon price $P(t, T)$ given by

$$P(t, T) = \tilde{E} \left[e^{\int_t^T r(s) ds} \middle| \mathcal{F}_t \right]. \quad (1.44)$$

where \tilde{E} denotes the conditional expectation with respect to the *risk-neutral probability* \tilde{P} (see section 1.6). From (1.44), it can be proved that $r(t)$ satisfies the following SDE:

$$dr(t) = a(r, t)dt + b(r, t)dW(t). \quad (1.45)$$

Expression (1.45) implies that $r(t)$ is a *Markov process*. We will discuss this model in more detail in the next section. See [5] and [24] for more details.

Forward rate models use the instantaneous forward rate $f(t, T)$ rather than the short rate. For $T > 0$ fixed, we just have to specify the SDE

$$df(t, T) = a(t, T)dt + \sigma dW(t), \quad (1.46)$$

where $\sigma > 0$ is constant and $a(t, T)$ is a deterministic function. The bond pricing formula in this case is given by a similar expression as (1.44).

Notice that the interest rate models given by (1.45), and (1.46) are related, since $r(t) = R(t, t) = f(t, t)$. In this thesis we will only consider short rate models, where the future behavior of interest is given by expression (1.45) with appropriate choices for $a(r, t)$ and $b(r, t)$.

1.5.3 Short Rate Models

Short rate models, where the short rate $r(t)$ is specified by (1.45), are a special cases of the more general *one-factor models*, where only one interest rate is specified to predict the future

behavior of interest rates. Another one-factor model is given by (1.46) where the only interest rate specified is $f(t, T)$.

We examine again equation (1.45):

$$dr(t) = a(r, t)dt + b(r, t)dW(t).$$

The term $W(t)$ is a *standard Wiener process* or *Brownian motion* under the *real-world measure* P , while $a(r, t)$ and $b(r, t)$ are “well-behaved” functions, and $\mathcal{F}_t = \sigma(\{W(s) : s \leq t\})$ is the *sigma-algebra* generated by the history of $W(s)$ up to time t (or as the “history” of to time t). For short rate models it is typically assumed that $a(r, t) = a(r)$ and $b(r, t) = b(r)$, so that the process is a Markov chain and time homogeneous (that is, $r(t)$ is a *stationary Markov chain*), this is a technical detail that is needed when proving some results of the theory of interest rate models. Such proofs fall beyond the scope of this work and will be omitted. See [5] and [4], however, for more details.

By explicitly specifying the two functions $a(r, t)$ and $b(r, t)$ we obtain a particular short rate model. Several of this choices are presented in table 1.1 (also see [5]).

Short rate models for $r(t)$		
Model	$a(r)$	$b(r)$
Merton (1973)	μ	σ
Dothan (1978)	μr	σr
Vasicek (1977)	$\alpha(\mu - r)$	σ
Cox-Ingersoll-Ross (1985)	$\alpha(\mu - r)$	$\sigma\sqrt{r}$
Pearson-Son (1994)	$\alpha(\mu - r)$	$\sigma\sqrt{r - \beta}$
Brennan-Schwartz (1979)	$\alpha(\mu - r)$	σr
Black-Karansinski (1991)	$\alpha r - \gamma r \log r$	σr

Table 1.1 Some short rate models from the literature

Among all the possible choices presented in table 1.1, there are a few desirable, but not essential, characteristics for an short rate model. The most basic of these desired characteristics are:

- Interest rates should be positive.
- $r(t)$ should be *mean-reverting* (also called *autoregressive*).

- We should get simple formulas for bond prices and for the prices of other interest rate contracts.

The first condition is desired of all interest rates. The second condition assumes that $r(t)$ cannot drift off to plus or minus infinity or to zero, but must eventually be pulled back to some long-term target. The third characteristic is a matter of computational convenience rather than of economic principle, the existence of elegant formula do not prove the worth of a model. Some, but not all, of the short rate models shown above in table 1.1 have a few or all of the three desired characteristics, see table 1.2 (also see [5]).

Key characteristics of standard short rate models			
Model	$r(t) \geq 0$?	Autoregressive?	Simple formulas?
Merton (1973)	No	No	Yes
Dothan (1978)	Yes	No	No
Vasicek (1977)	No	Yes	Yes
Cox-Ingersoll-Ross (1985)	Yes	Yes	Yes
Pearson-Son (1994)	Yes (if $\beta > 0$)	Yes	No
Brennan-Schwartz (1979)	Yes	Yes	No
Black-Karansinski (1991)	Yes	Yes ($\gamma > 0$)	No

Table 1.2 Characteristics of short rate models

Besides the three desirable characteristics presented above, there are other characteristics that a short rate model should satisfy. In general these models may fail these additional criteria, and for this reason models which incorporate more than one factor are often used.

There are two approaches to derive pricing formulas from a short rate model given by (1.45). One is called the *martingale approach*, and while the other is the *PDE approach*. The martingale approach uses the theory of martingales to establish prices and hedging strategies. The PDE approach started with [28], and is a general PDE approach very similar to the option pricing approach developed by the Black-Scholes equation from [2]. Although the martingale approach is generally thought to be more powerful and intuitive than PDE approach, the latter still provides us with a useful tool for the development of numerical methods and is the approach we follow in this thesis.

1.5.4 The Cox-Ingersoll-Ross Model

The *Cox-Ingersoll-Ross model* (usually abbreviated as CIR model) is a short rate model where the behavior of interest rates is given by (1.45) with the choice of $a(r, t) = a(r) = \alpha(\mu - r)$ and $b = b(r) = \sigma\sqrt{r}$, where α , μ and σ are positive constants. This model was introduced in [7] as an extension the model developed in [28]. It specifies that the short interest rate follows the following stochastic differential equation:

$$dr(t) = \alpha(\mu - r(t)) dt + \sigma\sqrt{r(t)} dW(t). \quad (1.47)$$

The *drift factor*, $\alpha(\mu - r_t)$, ensures the mean reversion of the interest rate towards the long run value μ , with speed of adjustment governed by the strictly positive parameter α . The standard *deviation or volatility factor*, $\sigma\sqrt{r_t}$, avoids the possibility of negative interest rates for all nonnegative values of α and μ . An interest rate of zero is also precluded if the condition

$$2\alpha\mu > \sigma^2$$

is met. More generally, when the interest rate $r(t)$ is close to zero, the standard deviation also becomes close to zero, which dampens the effect of the random shock on the rate. Consequently, when the rate gets close to zero, its evolution becomes dominated by the drift factor, which pushes the rate upwards, towards the equilibrium μ . See [4] for more details.

The CIR model given by (1.47) is the interest rate model adopted by most of the literature on mortgage valuation and will be adopted in this thesis as well, see subsection 2.2.1.

1.6 Risk-Neutral Valuation

We have seen in equations (1.45), (1.46), and (1.47) that there is always a term involving the standard Wiener process $W(t)$. It is this term that makes these stochastic, rather than ordinary, differential equations. We can informally think of this second term as a random perturbation, capturing the randomness of the financial markets as well as other unpredictable economic considerations. A standard Wiener process must be defined by means of a probability measure. When dealing with interest rates, there is a natural, "real-world" probability measure

P , precisely the one that arises due to the uncertainties of the financial markets and that measures the probability that an interest rate will take a particular value or values. This measure, however, is rather unknown and can only be inferred from historical values. Even though it may seem as if this a statistical analysis of historical values, from a mathematical finance point of view, we need much more, in particular we wish to guarantee that the resulting interest rates and pricing PDE do not contain any unknown functions. For this reason we need to characterize this unknown real-world probability measure. This is typically achieved by finding an equivalent probability measure, a *risk-neutral probability measure* \tilde{P} that is equivalent to P . We can then write the corresponding SDEs using a Wiener process defined by this measure \tilde{P} and derive pricing PDEs and formulas from there. This often results in additional terms in our equations that have to be determined before hand. One of these terms is the *market price of risk*, that appears in the derivation of valuation PDEs.

The basic tool used to derive risk neutral probability measures is the *Girsanov Theorem* from stochastic calculus. This theorem tells how stochastic processes change under changes in the underlying measure. It shows how to convert from the real-world measure P , to the risk-neutral measure \tilde{P} .

Theorem 1.6.1 (Girsanov). *Let $\{W(t) : 0 \leq t \leq T\}$ be a one dimensional Brownian motion on the probability space (Ω, P, \mathcal{F}) and let $\mathcal{F}_t^W = \sigma \{W(s) : 0 \leq s \leq t\}$ be the filtration generated by this Brownian motion. Let $\{\theta(s) : 0 \leq s \leq T\}$ be a stochastic process adapted to the filtration $\{\mathcal{F}_t^W\}_{0 \leq t \leq T}$ that satisfies*

$$E \left[e^{\int_0^T |\theta(s)|^2 ds} \right] < \infty.$$

Define

$$Z(t) = \exp \left(- \int_0^t \theta(s) dW(s) - \frac{1}{2} \int_0^t |\theta(s)|^2 ds \right), \quad 0 \leq t \leq T.$$

and consider the process

$$\tilde{W}(t) = W(t) + \int_0^t \theta(s) ds, \quad 0 \leq t \leq T.$$

Define the probability \tilde{P} on (Ω, \mathcal{F}_T) by

$$\frac{d\tilde{P}}{dP} = Z(T),$$

that is, for any $A \in \mathcal{F}_T$,

$$\tilde{P}(A) = E_P[Z(T) \cdot \chi_A].$$

Then

1. $\{Z(t)\}$, is a P -martingale with respect to $\{\mathcal{F}_t^W\}_{0 \leq t \leq T}$.
2. $\{\tilde{W}(t) : 0 \leq t \leq T\}$ is a \tilde{P} -Brownian motion.

A proof of this important theorem can be found in [24]. This risk-neutral probability \tilde{P} is also denoted by Q by many sources in the literature.

We now outline how the Girsanov Theorem can be used to derive the market price of risk. Assume first that the price of a stock is given the standard model described in (1.6)³:

$$\begin{aligned} dS(t) &= S(t) [\mu dt + \sigma dW(t)], \\ S(0) &= S_0, \end{aligned}$$

and that a bank account follows the dynamics given by $dB(t) = rB(t)dt$, $B(0) = 1$, where r denotes the constant risk-free interest rate of this account. We now look at the discounted process, that is to say we look at prices in *time-zero dollars*, then stock price satisfies the following SDE:

$$\begin{aligned} d(e^{-rt}S(t)) &= e^{-rt}S(t) [(\mu - r)dt + \sigma dW(t)] \\ &= \sigma e^{-rt}S(t) \left[\left(\frac{\mu - r}{\sigma} \right) dt + dW(t) \right], \end{aligned}$$

while the money market account is given by $e^{-rt}B(t) = 1$. Using now theorem 1.6.1, we define:

$$\tilde{W}(t) = W(t) + \int_0^t \left(\frac{\mu - r}{\sigma} \right) dt, \tag{1.48}$$

and let the probability \tilde{P} be defined as

$$\frac{d\tilde{P}}{dP} = Z(T), \tag{1.49}$$

³Notice that we slightly changed the notation from the one used in 1.3.1, to adapt it to the notation used in the current section

where $Z(t)$ from theorem 1.6.1 is given by

$$Z(t) = \exp \left[- \left(\frac{\mu - r}{\sigma} \right) W(t) - \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 t \right].$$

This new function $\widetilde{W}(t)$, is a new Brownian motion with respect to \widetilde{P} . Moreover, the discounted stock price also satisfies

$$d(e^{-rt}S(t)) = \sigma(e^{-rt}S(t)) d\widetilde{W}(t),$$

and hence it is a martingale with respect to \widetilde{P} , while the discount money market account $e^{-rt}B(t)$ is a constant martingale.

This new Wiener process given by (1.48) is used to derived valuation PDEs and formulas, as it guarantees that the involved interest rates and other related stochastic processes are martingales and Markov processes. The function $\lambda(t) = \frac{\mu - r}{\sigma}$ in (1.48) is the so-called market price of risk, and has an economic interpretation: it is the amount of extra return that the market requires in order to be compensated for taking some particular risk, as in classical economic theory, no rational person would invest unless he expects to beat the return from a risk-free asset.

The risk-neutral probability measure \widetilde{P} is indirectly defined by means of theorem 1.6.1 as a *Radon-Nikodym derivative*, the left-hand side of (1.49). In general there may be more than one risk-neutral measure. A risk-neutral measure (called *equivalent martingale measure*, or *Q-measure*) is a probability measure that results when one assumes that the current value of any financial asset is equal to the expected value of the future payoff of the asset discounted at the short interest rate. In mathematical analysis terms: Given the real-world probability P on the filtered space (Ω, \mathcal{F}) , with filtration $\{\mathcal{F}_t^W : 0 \leq t \leq T\}$ as before, we say that the probability \widetilde{P} is a risk-neutral probability if the following two conditions hold:

- \widetilde{P} and P are equivalent on \mathcal{F}_T (i.e. $P(A) = 0$ iff $\widetilde{P}(A) = 0$, for all $A \in \mathcal{F}_T$).
- For each tradable asset $S(t)$, the discounted process $\{e^{-rt}S(t) : 0 \leq t \leq T\}$ is a \widetilde{P} -martingale.

In the context of the risk-neutral probability, if the price of an asset satisfies the SDE

$$dY(t) = \mu(Y, t)dt + \sigma(Y, t)d\widetilde{W}(t),$$

then the price of a financial derivative contract U with underlying Y , is given by

$$U(y, t) = E \left[e^{-\int_0^T r(Y(s), s) ds} \Phi(Y(T)) \mid Y(t) = y \right],$$

where $\Phi(Y)$ is some known payoff function and r is the short interest rate.

We now finalize this section with a few further remarks regarding arbitrage and complete markets, two concepts that are taken for granted in the mortgage valuation literature. We define an *arbitrage portfolio* as a portfolio value process $\{X(t)\}$ that satisfies

- $P[X(T) \geq 0] = 1$,
- $P[X(T) > 0] > 0$.

If \tilde{P} is a risk-neutral probability, then $P[X(T) \geq 0] = 1$ implies that $\tilde{P}[X(T) \geq 0] = 1$, moreover, $P[X(T) > 0] = 0$ implies $\tilde{P}[X(T) > 0] = 0$, and hence $P[X(T) > 0] > 0$.

We can then state the following result, known in the mathematical finance literature as the *first fundamental theorem of asset pricing*:

Theorem 1.6.2. *In a market model, there are no arbitrage processes if and only if there is a risk-neutral probability measure \tilde{P} .*

Arbitrage-free markets are part of the assumptions of mortgage valuation.

Now, we define a *complete market*: A market is complete if each derivative asset can be *hedged*, that is, we can create a portfolio to "match" the value of the financial derivative at final time. This portfolio is called a *hedging or replicating portfolio*. This definition in turn takes us into the *second fundamental theorem of asset pricing*:

Theorem 1.6.3. *A market model with a risk-neutral probability \tilde{P} is complete if and only if \tilde{P} is unique.*

Hence, in the mortgage valuation setting, the mortgage market is assumed to be both, arbitrage-free and complete, and therefore the risk neutral probability is assumed to be unique.

CHAPTER 2. STANDARD OPTION-THEORETIC MORTGAGE VALUATION

In this chapter we review the standard mortgage valuation PDE models.

2.1 Mortgage Valuation Models

Having introduced the basic terminology in chapter 1 we now present an overview of some of the mortgage valuation models from the literature, and we also make some comments on option-theoretic models, the focus of this thesis.

2.1.1 Previous Literature

The models for mortgage valuation can be broadly divided into two categories. In the first category, the mortgage models look at values derived from a martingale approach and a statistical analysis, they are *forward in time*, since from the point of view of numerical analysis, future mortgage values are inferred from earlier values; typically the numerical simulations of these models use the Monte-Carlo method. In the second category we have mortgage values that are derived through PDEs, analogous to the Black-Scholes equation; these methods are *backward in time*, as they infer earlier mortgage values from a known terminal condition; the numerical simulations in these case typically involve finite difference approximations. These backward models visualize a mortgage value as having similar properties as an option, and therefore they are called *option-theoretic valuation models*. See [16] for an overview of these models.

During the early years of the study of mortgage valuation models, researchers considered simpler contracts that had some, but not all, of the conditions found in a mortgage contract,

see for instance [3] and [17]. Common approaches have been to concentrate on the right to prepay, ruling out the possibility of default, or consider default, ruling out the possibility of prepayment. Later studies, such as [17], suggested that such approach was rather incomplete, as a mortgage model must, at the very least, consider the possibility of early termination by means of either default or prepayment. Reasons for defaulting or prepayment are complex, and some models attempt to incorporate such complexity, for instance [15] and [26]. These authors point out, however, that termination of a mortgage contract may occur due to personal reasons, for instance a new job, divorce, death of a relative, forcing a homeowner to change residences. However, there are also financial reasons to terminate a mortgage contract and these apply equally well to all individual borrowers. Financial reasons play an important role in the case of default, while personal reasons are the main influence in the decision to prepay.

Several studies have found that there is a significant gap between the results obtained by a simplified contract and those of a full, real-world mortgage contract, see [25]. Option-theoretic models, however, still provide useful insights into the future values and behavior of mortgage values and they are widely studied. This is the type of model we consider in this work.

2.1.2 Option-Theoretic Models

Option-based pricing models have roots in early economic research, as far back as the work on stock options of the french mathematician Louis Bachelier in 1900. The modern breakthrough came with the work of Black and Scholes in [2]. These models identify two sources of uncertainty: the default risk and term structure or interest rate risk. Default risk is tied to the property price, which are traded assets. The term structure risk is much more complex, since interest rates are not directly traded. So, whereas for a property price we do not consider attitudes toward risk, for the term structure we must consider attitudes toward interest rate risk as well as the trend of interest rates movement.

Unlike the relatively simple Black-Scholes model from [2], where closed-form solutions actually exist, in the case of mortgage valuation models and due to the complexity of the mortgage contracts, there is really no hope for a simple valuation formula. Mortgages are, in fact, among

the most complex contracts ever devised, so to value them, most models resort to numerical methods to find an approximate value. This is indeed preferred, as [30] points out, we prefer to have an accurate value of the right model, rather than an analytical formula of the wrong one.

In the option-theoretic mortgage models, the right to prepay is typically regarded as call option, while the right to default is thought of as a put option. Despite the difference between theoretical and real mortgage values, the option approach often outperforms other models since it models the economic structure on which a borrower's behavior is based, and can adapt to significant changes in the contract terms or the economic environment.

As already mentioned, numerical methods for the valuation of option-theoretic mortgage models typically involve backward methods, which rely on the fact that the value of a mortgage is known at the time of maturity and, given the economic environment, one can work backward in time to find the value of a mortgage at a previous instant of time for all possible values of interest rate and house prices.

See [17], [16], [15], and [25] and the references within for more details regarding these type of models.

2.2 The Standard Option-Theoretic Mortgage Model

We now introduce this section the most commonly studied option-theoretic mortgage valuation model, from which the model of this thesis is derived. This approach regards a mortgage as consisting of three basic components: the value of the future promised payments to the lender, the value of the borrower's option to default, and the value of the borrower's option to prepay. As we mentioned before in chapter 1, the only type of mortgage considered in this thesis is the US fixed rate mortgage (FRM), where the interest rate of this contract is set at origination and remains constant thereafter.

More on this standard models can be found in [14], [17], [15], [16], and [25].

2.2.1 State Variables

In this modeling framework, we assume there are two sources of uncertainty: the default risk and the interest rate risk. The default risk is mainly tied to the random fluctuations of house prices. On the other hand, the interest rate risk is tied to random perturbations of financial markets and the behavior of the economy as a whole. For these reasons, we chose two specific models that will characterize these two risks.

In the standard model, the variable house price is modeled as a *log-normal stochastic process*, which is the solution to the following SDE:

$$dh = (r - \delta)hdt + \sigma_h h dX_h \quad (2.1)$$

where μ , δ , and σ_h are constant, X_h is a standard Wiener process, and t denotes time. Notice that this is essentially the same as equation (1.6), with minor changes in the parameters and the notation. Indeed, the house price is modeled in the same way we modeled the price of an asset in 1.3.1 and the assumptions that lie under equation (2.1) are the same as those for (1.6).

The term structure of interest rates is modeled by the Cox-Ingersoll-Ross (CIR) model presented in 1.5.4, where the spot interest rate r is driven by a mean-reverting square root process, that is to say it satisfies the following SDE:

$$dr = \kappa(\theta - r)dt + \sigma_r \sqrt{r} dX_r \quad (2.2)$$

where κ , θ , and σ_r are constants, X_r is a standard Wiener process, and t denotes time. Notice as well, that this is exactly the same as equation (1.47) with some change in the notation, for instance we will now write r rather than $r(t)$ and etc.¹.

We remark that equations (2.1) and (2.2) are coupled by the inclusion of r in the drift term $(r - \delta)hdt$ of the equation for h . This is due to the risk-neutral reasoning presented in section 1.6. In particular, the market price of risk is regarded as having been absorbed into the statistical estimation of the reversion and long-term average parameters, κ and θ respectively, in the interest rate equation (2.2). Furthermore, the arbitrage-free pricing theory from 1.5.1

¹This departure from the notation used in chapter 1 is done to be consistent with the notation presented in the mortgage valuation literature such as [15] and [25].

requires that this market price of risk also disappear, and hence r must be part of the equation for h as well. In addition to this relationship between (2.1) and (2.2), the Wiener processes may be correlated. We denote by ρ the correlation coefficient between X_h and X_r .

These two models for the house price h and the short interest rate r , are assumed to capture all the sources of uncertainty in the standard approach to mortgage valuation, therefore h and r are adopted as *state variables*, and the mortgage value will depend on them as well as on time t .

2.2.2 Definitions, Temporal, and Boundary Conditions

To give a precise mathematical meaning to the standard model, we present here some notation and several definitions for a few of the quantities involved in a mortgage contract. See section 1.2 for an explanation of these terms.

We start with the following definitions that will be used throughout this thesis:

- L : Original loan.
- c : Fixed yearly mortgage contract interest rate.
- N_m : Maturity of the loan in months.
- i : Payment date month, $1 \leq i \leq N_m$.
- T_i : i -th payment date in years, i.e. $T_i = \frac{i}{12}$, $1 \leq i \leq N_m$. Also set $T_0 := 0$, so T_i is defined for $0 \leq i \leq N_m$.
- MP : Fixed monthly mortgage payment (See appendix A for a derivation),

$$MP = \frac{L \left(1 + \frac{c}{12}\right)^{N_m} \left(\frac{c}{12}\right)}{\left(1 + \frac{c}{12}\right)^{N_m} - 1}. \quad (2.3)$$

- $PB(i)$: Unpaid principal balance after the i -th payment date (See appendix A for a derivation),

$$PB(i) = \frac{L \left[\left(1 + \frac{c}{12}\right)^{N_m} - \left(1 + \frac{c}{12}\right)^i \right]}{\left(1 + \frac{c}{12}\right)^{N_m} - 1}, \quad (2.4)$$

for $0 \leq i \leq N_m - 1$.

- $TD(t)$: Total debt at time t , i.e. unpaid principal plus accrued interest for $T_i < t \leq T_{i+1}$

$$TD(t) = \left[1 + c(t - \tau(i))\right] PB(i). \quad (2.5)$$

In this model no prepayment penalty considered, and hence the total debt (2.5) is slightly different than the one in [25].

In addition to the definitions above, we will use the following notation for the different mortgage components²:

- $A(r, t; i)$: Value at time t of the promised mortgage payments, from payment date i to N_m .
- $D(h, r, t; i)$: Value at time t of the default option, when the next mortgage payment is due at time T_i .
- $C(h, r, t; i)$: Value at time t of the prepayment option, when the next mortgage payment is due at time T_i .
- $V(h, r, t; i)$: Value at time t of the contract, when the next mortgage payment is due at time T_i .

As we already pointed out, in the standard model the default and prepayment features of a mortgage are modeled as options. Default is modeled as a sequence of linked monthly European put options, while prepayment is modeled as an American call option. These European options start right after a monthly payment is made at time T_{i-1} and have expiration at the next payment date T_i . The American option starts at the origination of the mortgage at time T_0 and has the same expiration time as the mortgage itself at time T_{N_m} . With the notation as introduced above, $D(h, r, t; i)$ and $C(h, r, t; i)$ denote the value of the European and American option within the time interval $(T_{i-1}, T_i]$.

The standard model assumes that the value of the mortgage is the same as scheduled payments, minus the value of the borrower's options to terminate the mortgage, that is to say:

$$V(h, r, t; i) = A(r, t; i) - D(h, r, t; i) - C(h, r, t; i). \quad (2.6)$$

²The arguments of these functions will be dropped whenever it is convenient and there is no risk of confusion.

The problem consists of finding the value of the mortgage $V(h, r, t; i)$, and in principle its components $A(r, t; i)$, $D(h, r, t; i)$, and $C(h, r, t; i)$, for all possible house prices h , all possible values of the short interest rate r , and all times before maturity t . As we have a sequence of monthly European options, this problem is solved month by month.

In this thesis, we will concentrate on finding the value of the mortgage $V(h, r, t; i)$ and will not be concerned with $A(r, t; i)$, $D(h, r, t; i)$, and $C(h, r, t; i)$, which can be found in a similar fashion, see [25] for more on this. Summarizing, we are interested in finding $V(h, r, t; i)$ on the domain given by $0 \leq h < \infty$, $0 \leq r < \infty$, and $T_{i-1} < t \leq T_i$ for $i = 1, \dots, N_m$.

In addition to the no-arbitrage considerations already included in the parameters of equations (2.1) and (2.2), the mortgage value must also satisfy another arbitrage-free condition. The contract rate must be so that the *equilibrium condition*

$$V(h, r, 0; 1, c) = (1 - fee)L \quad (2.7)$$

holds. In (2.7), the quantity *fee* is the mortgage fee, a charge made to the borrower for the origination of a mortgage. Its value is given as a percentage.

Now, at the maturity of the mortgage, when $t = T_{N_m}$, we completely know the possible values a mortgage and its components may take, as in that case a borrower may either default only or just pay the last remaining monthly payment. These constitute the following final conditions:

$$A(T_{N_m}; N_m) = MP, \quad (2.8)$$

$$V(T_{N_m}; N_m) = \min\{MP, h\}, \quad (2.9)$$

$$C(T_{N_m}; N_m) = 0, \quad (2.10)$$

$$D(T_{N_m}; N_m) = \max\{0, MP - h\}. \quad (2.11)$$

Moreover, we also know the value of the mortgage and its components at the earlier payment dates, when $t = T_i$ for $1 \leq i < N_m$, since the default component is a European put option and whether a borrower would default or not is the question of whether the option would be exercised or not. These payment day conditions also link the different European options across

monthly payments. These conditions:

$$A(T_i; i) = A(T_i+; i+1) + MP, \quad (2.12)$$

$$V(T_i; i) = \min\{V(T_i+; i+1) + MP, h\}, \quad (2.13)$$

$$C(T_i; i) = C(T_i+; i+1), \quad \text{if no default}, \quad (2.14)$$

$$C(T_i; i) = 0, \quad \text{if default}, \quad (2.15)$$

$$D(T_i; i) = D(T_i+; i+1), \quad \text{if no default}, \quad (2.16)$$

$$D(T_i; i) = A(T_i; i) - h, \quad \text{if default}. \quad (2.17)$$

Notice we use notation $A(T_i+; i+1) := \lim_{t \rightarrow T_i^+} A(t; i+1)$, and etc. The payment date conditions essentially answer the question of how to define the final conditions for a mortgage component at the end of a month time period, when we use the information from the next month, that is information from the interval $(T_i, T_{i+1}]$.

Conditions (2.8)-(2.11) and (2.12)-(2.17), identify two kinds of situations regarding default on the mortgage. Default implies that $V(T_i; i) = V(T_i+; i+1) + MP$, while on the other end, no default implies that $V(T_i; i) = h$. Prepayment does not directly influence the situation at the payment dates, as the American out option does not expire until the final time T_{N_m} , instead it makes itself evident through a free boundary condition, see (2.32) below.

Finally, to complete the description of the standard valuation model we must consider boundary values, which arise when we consider extreme values of the house price h and the interest rate r . On this respect there seem to be not a complete agreement among the literature, especially for the cases when h is very large or r very small. We present here the boundary conditions used by some of the recent literature, such as [25], for a slightly different choice of conditions see [15].

We start with boundary conditions for the promised future payments A , since this is a function that is independent of h :

- at $r = 0$,

$$A(0, t) = \frac{L}{N_m}. \quad (2.18)$$

- as $r \rightarrow \infty$,

$$\lim_{r \rightarrow \infty} A(r, t) = 0. \quad (2.19)$$

Next, for the other mortgage components, including the mortgage value itself, we have the following boundary conditions:

- at $h = 0$,

$$D(0, r, t) = A(r, t), \quad (2.20)$$

$$C(0, r, t) = 0, \quad (2.21)$$

$$V(0, r, t) = 0, \quad (2.22)$$

- as $h \rightarrow \infty$,

$$\lim_{h \rightarrow \infty} D(h, r, t) = 0, \quad (2.23)$$

$$\lim_{h \rightarrow \infty} \frac{\partial C}{\partial h}(h, r, t) = 0, \quad (2.24)$$

$$\lim_{h \rightarrow \infty} \frac{\partial V}{\partial h}(h, r, t) = 0, \quad (2.25)$$

- at $r = 0$,

$$D(h, 0, t) = 0, \quad (2.26)$$

$$C(h, 0, t) = 0, \quad (2.27)$$

$$V(h, 0, t) = A(r, t), \quad (2.28)$$

- as $r \rightarrow \infty$,

$$\lim_{r \rightarrow \infty} D(h, r, t) = 0, \quad (2.29)$$

$$\lim_{r \rightarrow \infty} C(h, r, t) = 0, \quad (2.30)$$

$$\lim_{r \rightarrow \infty} V(h, r, t) = 0. \quad (2.31)$$

Next, in principle before a borrower continues making the scheduled monthly payments, he must make sure that a mortgage has a value that is smaller than the total debt (2.5), otherwise

it is not advantageous for him and it would be better for him to prepay the mortgage by exercising the prepayment American put option. This is mathematically expressed as a free boundary condition, where the prepayment option is not exercise if the following inequality holds:

$$V(h, r, t) \leq TD(t), \quad (2.32)$$

for all times $0 \leq t \leq T_{N_m}$. The option is exercised and prepayment first occurs as soon as $V(h, r, t) = TD(t)$. This constraint (2.32) creates two regions in the plane of the state variables h and r in a manner similar to that already described in section 1.4.

The temporal condtions, together with the boundary conditions (2.18), (2.19), (2.20)-(2.22), (2.23)-(2.25), (2.26)-(2.28) and (2.29)-(2.31) and the free boundary condition (2.32), are all necessary to find the value of a mortgage and its related components. The only ingredient that we are still missing in order to close the standard mortgage model is a valuation PDE, which will be derived in section 2.3.

2.2.3 Time Reversal

Before we move on to the next section, we will describe a small change of variable, which is just a change in the direction of time. This modification is not essential, but it simplifies the numerical schemes that we will use in chapter 5 to approximate the solution to the mortgage value. Let

$$\tau = T_{N_m} - t. \quad (2.33)$$

We will now denote the payment dates now by τ_k and will define them to be:

$$\tau_{k+1} = T_{N_m} - T_{N_m-k},$$

where $k = 0, 1, \dots, N_m - 1$, and moreover, we will also set $\tau_{N_m+1} = T_{N_m}$.

With this change (2.33) and the change in the notation, the mortgage components are now defined on the interval $[\tau_k, \tau_{k+1})$, for instance, $A(r, \tau; k)$ only makes sense for $\tau_k \leq \tau < \tau_{k+1}$ and etc.

There are few changes in the functions defined in the previous section, for instance total debt (2.5) changes to:

$$TD(\tau) = \left[1 + c(\tau_{k+1} - \tau)\right] PB(N_m - k), \quad (2.34)$$

where $\tau_k \leq \tau < \tau_{k+1}$ and $1 \leq k \leq N_m$. Also, the final time conditions (2.8)-(2.11) now correspond to $\tau = \tau_1 = 0$ and they become:

$$A(\tau_1; 1) = MP, \quad (2.35)$$

$$V(\tau_1; 1) = \min\{MP, h\}, \quad (2.36)$$

$$C(\tau_1; 1) = 0, \quad (2.37)$$

$$D(\tau_1; 1) = \max\{0, MP - h\}. \quad (2.38)$$

The payment date conditions (2.12)-(2.17) now correspond to times $\tau = \tau_{k+1}$, where $1 \leq k \leq N_m - 1$, and are given by:

$$A(\tau_{k+1}; k+1) = A(\tau_{k+1}-; k) + MP, \quad (2.39)$$

$$V(\tau_{k+1}; k+1) = \min\{V(\tau_{k+1}-; k) + MP, h\}, \quad (2.40)$$

$$C(\tau_{k+1}; k+1) = C(\tau_{k+1}-; k), \quad \text{if no default}, \quad (2.41)$$

$$C(\tau_{k+1}; k+1) = 0, \quad \text{if default}, \quad (2.42)$$

$$D(\tau_{k+1}; k+1) = D(\tau_{k+1}-; k), \quad \text{if no default}, \quad (2.43)$$

$$D(\tau_{k+1}; k+1) = A(\tau_{k+1}; k+1) - h, \quad \text{if default}. \quad (2.44)$$

Notice again that we use the notation $A(\tau_{k+1}-; k) := \lim_{\tau \rightarrow \tau_{k+1}^-} A(\tau; k)$, and etc.

The other remaining quantities introduced in 2.2.2 do not change and remain the same.

2.3 Valuation PDE

One of the main advantages of option-theoretic mortgage models over other alternatives is that we can derive a partial differential equation whose solution is precisely the mortgage value V . This PDE is analogous to the Black-Scholes PDE (1.12) and it is inferred by similar arguments as those used in 1.3.2.

In this section we present a short outline of the derivation of this valuation PDE. We concentrate our attention on the mortgage value V , but we notice that the other mortgage components such as the values D and C also satisfy the same PDE, but with different temporal and boundary conditions. The mortgage component A is a slightly different case and, just to complete our discussion, we present the derivation of another PDE that is analogous to the bond pricing PDE, see for instance [30] for more details on bond pricing.

2.3.1 Derivation of the Valuation PDE

Recall that the mortgage value V depends on the house price h , the interest rate r , and the time t . We will use the ideas and the notation of in stochastic calculus, and we start by looking for SDE for V ; these SDE will contain some random terms, and thus our ultimate goal is to get rid of these terms and arrive to a purely deterministic PDE.

Using the Ito's lemma from stochastic calculus to derive the following expression for the increment dV :

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial h}dh + \frac{\partial V}{\partial r}dr + \frac{1}{2} \left(\frac{\partial^2 V}{\partial h^2}dh^2 + 2\frac{\partial^2 V}{\partial h \partial r}dhdr + \frac{\partial^2 V}{\partial r^2}dr^2 \right) + \dots \quad (2.45)$$

Ito's lemma can be thought of as a generalization of the chain rule in the context of stochastic processes, and (2.45) can be visualized as a stochastic version of a Taylor series expansion, for more details on this powerful mathematical tool see [24]. Now, It is a fact from stochastic calculus as well, that if W is a standard Wiener process, then

$$dW^2 \rightarrow dt \quad \text{as} \quad dt \rightarrow 0,$$

$$dWdt \rightarrow o(dt).$$

This, together with equations (2.1) and (2.2) that:

$$dh^2 \rightarrow \sigma_h^2 h^2 dX_h^2 \rightarrow \sigma_h^2 h^2 dt, \quad (2.46)$$

$$dr^2 \rightarrow \sigma_r^2 r dX_r^2 \rightarrow \sigma_r^2 r dt, \quad (2.47)$$

$$dhdr \rightarrow \sigma_h \sigma_r h \sqrt{r} dX_h dX_r = \rho \sigma_h \sigma_r h \sqrt{r} dt. \quad (2.48)$$

Moreover, the Wiener processes X_h and X_r satisfy the following relation:

$$dX_h dX_r = \rho dt. \quad (2.49)$$

Expressions (2.46)-(2.49) allows us to truncate the infinite series given by (2.45), and therefore we have:

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial h} dh + \frac{\partial V}{\partial r} dr + \frac{1}{2} \left(\sigma_h^2 h^2 \frac{\partial^2 V}{\partial h^2} + 2\rho\sigma_h\sigma_r h\sqrt{r} \frac{\partial^2 V}{\partial h\partial r} + \sigma_r^2 r \frac{\partial^2 V}{\partial r^2} \right) dt. \quad (2.50)$$

Next, we construct a portfolio Π , which consists of one mortgage $V_1(h, r, t)$, that has maturity T_1 , $-\Delta_1$ units of another mortgage $V_2(h, r, t)$, that has maturity T_2 , and $-\Delta_2$ units of a house with price h . This portfolio has value

$$\Pi = V_1 - \Delta_1 V_2 - \Delta_2 h. \quad (2.51)$$

We assume that Δ_1 and Δ_2 in (2.51) are constant over the small time period t to $t + dt$. The change in the portfolio over this time period is given by:

$$d\Pi = dV_1 - \Delta_1 dV_2 - \Delta_2 (dh + \delta h dt), \quad (2.52)$$

where δ is one of the parameters in (2.1). The risk in expression (2.52) vanishes when the random components of the terms dh and dr are eliminated. This can be achieved with a careful choice of the quantities Δ_1 and Δ_2 , in this case we choose them to be defined as:

$$\Delta_1 = \frac{\frac{\partial V_1}{\partial r}}{\frac{\partial V_2}{\partial r}}, \quad (2.53)$$

$$\Delta_2 = \frac{\partial V_1}{\partial h} - \Delta_1 \frac{\partial V_2}{\partial h}. \quad (2.54)$$

With the choices (2.53) and (2.54) for Δ_1 and Δ_2 , $d\Pi$ in (2.52) becomes:

$$\begin{aligned} d\Pi = & \frac{\partial V_1}{\partial t} dt + \frac{1}{2} \left(\sigma_h^2 h^2 \frac{\partial^2 V_1}{\partial h^2} + 2\rho\sigma_h\sigma_r h\sqrt{r} \frac{\partial^2 V_1}{\partial h\partial r} + \sigma_r^2 r \frac{\partial^2 V_1}{\partial r^2} \right) dt - \delta h \frac{\partial V_1}{\partial h} dt \\ & - \frac{\frac{\partial V_1}{\partial r}}{\frac{\partial V_2}{\partial r}} \left[\frac{\partial V_2}{\partial t} dt + \frac{1}{2} \left(\sigma_h^2 h^2 \frac{\partial^2 V_2}{\partial h^2} + 2\rho\sigma_h\sigma_r h\sqrt{r} \frac{\partial^2 V_2}{\partial h\partial r} + \sigma_r^2 r \frac{\partial^2 V_2}{\partial r^2} \right) dt - \delta h \frac{\partial V_2}{\partial h} dt \right]. \end{aligned} \quad (2.55)$$

Now, we make the standard arbitrage-free assumption, that the return on the portfolio Π given by (2.51) and the return on a riskless bank account is the same. This is analogous to (1.11). We then have that:

$$d\Pi = r\Pi dt. \quad (2.56)$$

Next, from the choices for Δ_1 and Δ_2 made in (2.53) and (2.54), we have that (2.51) becomes

$$\Pi = V_1 - \frac{\frac{\partial V_1}{\partial r}}{\frac{\partial V_2}{\partial r}} \cdot V_2 - \frac{\partial V_1}{\partial h} h + \frac{\frac{\partial V_1}{\partial r}}{\frac{\partial V_2}{\partial r}} \cdot \frac{\partial V_2}{\partial h} h,$$

and then, this expression combined together with (2.55) yields

$$\begin{aligned} & \frac{1}{\frac{\partial V_1}{\partial r}} \left[\frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma_h^2 h^2 \frac{\partial^2 V_1}{\partial h^2} + \rho \sigma_h \sigma_r h \sqrt{r} \frac{\partial^2 V_1}{\partial h \partial r} + \frac{1}{2} \sigma_r^2 r \frac{\partial^2 V_1}{\partial r^2} + (r - \delta) h \frac{\partial V_1}{\partial h} - r V_1 \right] \\ &= \frac{1}{\frac{\partial V_2}{\partial r}} \left[\frac{\partial V_2}{\partial t} + \frac{1}{2} \sigma_h^2 h^2 \frac{\partial^2 V_2}{\partial h^2} + \rho \sigma_h \sigma_r h \sqrt{r} \frac{\partial^2 V_2}{\partial h \partial r} + \frac{1}{2} \sigma_r^2 r \frac{\partial^2 V_2}{\partial r^2} + (r - \delta) h \frac{\partial V_2}{\partial h} - r V_2 \right]. \end{aligned} \quad (2.57)$$

Notice that the left-hand-side of this expression is a function of T_1 only, but not of T_2 , while the right-hand-side is a function of T_2 only, but not of T_1 . The only way this can happen is if both sides are independent of the maturity dates T_1 and T_2 . Hence, the left hand side of (2.57) is equal to some function that depends on the variables h , r and t only. Removing the subscript from V we then have the following equation

$$\frac{1}{\frac{\partial V}{\partial r}} \left[\frac{\partial V}{\partial t} + \frac{1}{2} \sigma_h^2 h^2 \frac{\partial^2 V}{\partial h^2} + \rho \sigma_h \sigma_r h \sqrt{r} \frac{\partial^2 V}{\partial h \partial r} + \frac{1}{2} \sigma_r^2 r \frac{\partial^2 V}{\partial r^2} + (r - \delta) h \frac{\partial V}{\partial h} - r V \right] = a(h, r, t), \quad (2.58)$$

where $a(h, r, t)$ is actually a market price of risk that can be calculated in a similar fashion as the one shown in section 1.6. This standard market price of risk the we take from the literature (see for instance [25]) is given by the expression

$$a(h, r, t) = -\kappa(\theta - r). \quad (2.59)$$

Finally, with the choice for a given by (2.59) above, the PDE for $V(h, r, t)$ is a consequence of equation (2.58). The valuation PDE is then

$$\begin{aligned} & \frac{1}{2} h^2 \sigma_h^2 \frac{\partial^2 V}{\partial h^2} + \rho h \sqrt{r} \sigma_h \sigma_r \frac{\partial^2 V}{\partial h \partial r} + \frac{1}{2} r \sigma_r^2 \frac{\partial^2 V}{\partial r^2} + \kappa(\theta - r) \frac{\partial V}{\partial r} \\ & + (r - \delta) h \frac{\partial V}{\partial h} + \frac{\partial V}{\partial t} - r V = 0. \end{aligned} \quad (2.60)$$

When we consider the reverse time given by (2.33), equation (2.60) changes slightly:

$$\begin{aligned} & \frac{1}{2} h^2 \sigma_h^2 \frac{\partial^2 V}{\partial h^2} + \rho h \sqrt{r} \sigma_h \sigma_r \frac{\partial^2 V}{\partial h \partial r} + \frac{1}{2} r \sigma_r^2 \frac{\partial^2 V}{\partial r^2} + \kappa(\theta - r) \frac{\partial V}{\partial r} \\ & + (r - \delta) h \frac{\partial V}{\partial h} - \frac{\partial V}{\partial \tau} - r V = 0. \end{aligned} \quad (2.61)$$

This small change is very important, as the equation (2.60) is a backward in time PDE, while equation (2.61) is a forward in time PDE. This forward version is easier to simulate and it will be used to derive the numerical scheme of chapter 5 and to perform the numerical simulations presented in section 5.4.

2.3.2 The Special Case of A

The value of the promised mortgage payments A depends only on the interest rate r and time t , it does not depend at all on the house price h . For this reason, A satisfies a different valuation PDE. We present the derivation of this equation, which as already said is similar to a bond pricing equation.

Similarly as in the previous section, start by setting up a portfolio consisting of two financial derivatives with a bond structure like A , but of different maturities, T_1 and T_2 . Denote the prices of these contracts by A_1 and A_2 respectively. The portfolio contains one unit of A_1 and $-\Delta$ units of the A_2 . Thus, the value of this portfolio is given by

$$\Pi = A_1 - \Delta A_2. \quad (2.62)$$

The change of this portfolio in a time period of length dt can be found by using Ito's lemma, like the process leading to (2.55) in the previous section. Hence,

$$d\Pi = \frac{\partial A_1}{\partial t}dt + \frac{\partial A_1}{\partial r}dr + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 A_1}{\partial r^2}dt - \Delta \left(\frac{\partial A_2}{\partial t}dt + \frac{\partial A_2}{\partial r}dr + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 A_2}{\partial r^2}dt \right). \quad (2.63)$$

Making the choice for Δ as

$$\Delta = \frac{\frac{\partial A_1}{\partial r}}{\frac{\partial A_2}{\partial r}}, \quad (2.64)$$

eliminates the random component of $d\Pi$ in (2.63), which then becomes

$$\begin{aligned} d\Pi &= \left(\frac{\partial A_1}{\partial t} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 A_1}{\partial r^2} - \frac{\frac{\partial A_1}{\partial r}}{\frac{\partial A_2}{\partial r}} \left(\frac{\partial A_2}{\partial t} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 A_2}{\partial r^2} \right) \right) dt \\ &= r \left(A_1 - \frac{\frac{\partial A_1}{\partial r}}{\frac{\partial A_2}{\partial r}} A_2 \right) dt. \end{aligned} \quad (2.65)$$

Next, the standard arbitrage-free argument is given again by:

$$d\Pi = r\Pi dt. \quad (2.66)$$

Combining (2.65) and (2.66) into one equation, and then collecting all the A_1 terms one side and all the A_2 on the other side, we obtain the expression:

$$\frac{\frac{\partial A_1}{\partial t} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 A_1}{\partial r^2} - r A_1}{\frac{\partial A_1}{\partial r}} = \frac{\frac{\partial A_2}{\partial t} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 A_2}{\partial r^2} - r A_2}{\frac{\partial A_2}{\partial r}}. \quad (2.67)$$

The left-hand side of (2.67) is a function of T_1 but not of T_2 , while the right-hand-side is a function of T_2 but not of T_1 , and therefore the only possibility is for both sides to be independent of the maturity date. Thus, dropping the subscript we have:

$$\frac{\frac{\partial A}{\partial t} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 A}{\partial r^2} - r A}{\frac{\partial A}{\partial r}} = a(r, t). \quad (2.68)$$

As with the market price of risk given by (2.59) from the previous subsection, there is a standard choice for the function $a(r, t)$:

$$a(r, t) = -\kappa(\theta - r). \quad (2.69)$$

From (2.68) and (2.69) we conclude that the valuation PDE for $A(r, t)$ is then

$$\frac{\partial A}{\partial t} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 A}{\partial r^2} + \kappa(\theta - r) \frac{\partial A}{\partial r} - r A = 0. \quad (2.70)$$

2.4 The Relevance of the Interest Rate Model in Mortgage Valuation

We conclude this chapter with a few remarks regarding the interest rate model given by (2.2). We may rightfully fear that the choice of interest rate model is not appropriate, and we may wonder what would be the effect in the mortgage model presented here, had we selected a different model instead of the CIR. It turns out, however, that such fears are unfounded.

In [6], the authors compare several widely used interest rate models, including a discrete time model, and explore how mortgage prices change depending on the choice of interest rate model. Their discoveries include that mortgage prices do not change significantly, and point out that the choice of interest rate model is more a matter of convenience than of accuracy. The authors point out that the ability to estimate model parameters is the most important characteristic of a sound interest rate process when the goal is that of modeling mortgages. According to [6], among the several well-known interest rate models, the CIR model from [7]

used here and the Dothan's log-normal model from [9] are the most widely used for mortgage-related studies. Both of these models guarantee nonnegative interest rates, so they are indeed desirable candidates for a short term interest rate model.

Thus, the choice of equation (2.2) as a interest rate model is not only standard, but also appropriate, and hence we adopt it in this thesis.

CHAPTER 3. MORTGAGE VALUATION WITH DISCOUNT FACTORS, COHERENT BOUNDARY CONDITIONS, AND STOCHASTIC TERMS

In this chapter we discuss the discount factors, the coherent boundary conditions, and the stochastic terms.

3.1 Discount Factors

Standard mortgage models like the one presented in the previous chapter are good approximations to real world situations, especially when the house price h increases and the interest rates are somehow high, as it was the case before the housing market crisis that started in 2007. However, when the house price h drops, the standard models are not very good approximations, as several situations that are not considered in these models may sudden occur and significantly affect the value of a mortgage. It is no longer reasonable to assume that the real world randomness are completely summarized by equations (2.1) and (2.2).

Thus to account for unforeseen scenarios that the standard mortgage modeling approach does not consider, this thesis incorporates a discount factor that will, in general, decrease the value of the mortgage depending on the most current economic conditions available. This discount factor is carefully chosen so as to include up-to-date information on several aspects of the economy. Therefore, mortgage values calculated with this approach will be more realistic and accurate.

3.1.1 Discounted Value of the Mortgage

The mortgage value is now modeled by "discounting" the mortgage value presented in (2.6):

$$V(h, r, t; i) = [A(r, t; i) - D(h, r, t; i) - C(h, r, t; i)] \cdot \lambda(h, r, t; i), \quad (3.1)$$

where the *discount factor* $\lambda(h, r, t; i)$ is a function that satisfies

$$0 \leq \lambda(h, r, t; i) \leq 1,$$

for $1 \leq i \leq N_m$. In general, this function is a function of the house price h , the short interest rate r , and time t , nevertheless, in practice the only explicit dependence is on time, so in this thesis we will assume that $\lambda(h, r, t; i) = \lambda(t; i)$, and that it is defined on the interval $(T_{i-1}, T_i]$, where $1 \leq i \leq N_m$.

The factor λ is further decomposed into three more components:

$$\lambda = \lambda_{cr} \cdot \lambda_{liq} \cdot \lambda_{fin}, \quad (3.2)$$

where, λ_{fin} is a factor tied to the conditions of the economy, λ_{liq} is a factor tied to the liquidity of the mortgage market, and λ_{cr} is a factor tied to the quality of the mortgage.

We make very special choices for these three discount factors λ_{fin} , λ_{liq} , and λ_{cr} . To define them, we make use of several economic indicators, mostly financial indices, which are in general weighted averages of selected economic quantities. For information about the indicators that will be presented in the next sections and to see some other possible choices, see [1].

3.1.2 Conditions of the Economy: $\lambda_{fin}(t)$

To define the factor $\lambda_{fin}(t)$ that will provide information about the general state of the economy, we will use the *Index of Leading Economic Indicators (LEI Index)*, which is designed to predict the economy's direction. This index is a composite of a select group of economic statistics that are known to swing up or down well in advance of the rest of the economy. This indicator is released monthly by The Conference Board, a private business research group. The report is published three weeks after the end of the reporting month. Revisions to this

index are usually minor, but more significant at other times. It is available on the Internet at <http://www.globalindicators.org>.

The LEI index is a composite index (where 1992 = 100) and it is made of ten components, seven nonfinancial and three financial, each with its own relative weight in the index¹:

- *Average hourly workweek in manufacturing* (weight: 25.4%): Taken from the employment report. A sustained rise or fall in the number of hours worked is often a telling sign of whether companies will soon hire or fire workers.
- *Average weekly initial claims for unemployment* (weight: 3.3%): Taken from the jobless claims report. It is one of the most sensitive to changing business conditions. Initial claims for unemployment benefits climb when the economic climate deteriorates, while it falls when the economy grows stronger.
- *Manufacturer's new orders for consumer goods and materials* (weight: 7.5%): Taken from the factory orders report. This inflation-adjusted statistic measures how comfortable manufacturers are with current inventory levels and projections of future consumer demand.
- *Vendor performance, or delivery times index* (weight: 7%): Taken from the Institute for Supply Management's manufacturing survey. If it takes longer to deliver products to consumers, this suggests that orders are flooding in so quickly that they are creating bottlenecks and products cannot be shipped as fast. On the other hand, quicker deliveries are more closely associated with an economic slowdown. As orders drop, a production crunch is less likely, and a turnaround time between order and delivery becomes shorter.
- *Manufacturer's new orders for nondefense capital goods* (weight: 1.9%): Taken from the factory orders report. Companies are less likely to spend on new capital equipment and goods if they suspect a business slowdown is looming.

¹As we pointed out above, an index such as the LEI index is a weighted average. It is typical in the economics literature to give these weights as percentages.

- *Building permits for new private homes* (weight: 2.7%): Taken from the housing starts release. Because most builders have to file for a permit to begin construction on private homes, tracking changes in the number of permits is a good indicator of future building activity.
- *Index of consumer expectations by the University of Michigan* (weight: 2.9%): Changes in expectations about future economic conditions and household income can alter consumer spending behavior.
- *Stock prices based on the S&P 500 stock index* (weight: 3.8%): The stock market has historically been a good leading indicator of economic turning points, as stocks today are priced to reflect expected earnings. A rise or fall in the S&P stock index is a barometer of what investors believe the economy will do in the future.
- *M2 money supply on real (inflation-adjusted) terms* (weight: 35.3%): Taken from the money supply figures from the Federal Reserve. M2 is one of the broader measures of the money supply and includes currency, demand deposits, saving accounts, and banks CDs. When M2 growth fails to keep pace with inflation, it is a sign that bank lending is slipping and the economy will soon weaken.
- *Interest rate spread between the 10-year Treasury bond and the federal funds rate* (10.2%): The difference between long-term rates and the federal funds rate (overnight borrowing rates by banks) has the best track record of the ten components for forecasting economic activity. This is why it has been given a relatively high weight in the index. If the spread (difference) in rates increases so that long-term rates become materially higher than short-term rates, it is a sign that the economy is on a growth path. However, if the spread narrows to the point where either there is no difference between the two maturities or they are inversely related (with short-term rates higher than long-term rates), it is indicative of an economy headed for trouble. This can happen when the Federal Reserve has driven short-term rates so high that the bond market is convinced economic activity will greatly weaken and bring down inflation with it.

The LEI index has the following limitations:

- The LEI index has successfully predicted recessions in the past, however it has also declined on numerous occasions without a corresponding downturn in the economy. Thus, the index can then give off false alarms about an oncoming recession too. It has a better track record of indicating when the economy is ready to emerge from a recession.
- Every month three out of the ten components underlying the LEI index have to be estimated: manufacturer's new orders for consumer goods and materials, manufacturer's new orders for non-defense capital goods, and the personal consumption deflator (used to calculate the "real" M2 money supply). Because of these estimates, the index may be substantially revised if the underlying figures turn out to be radically different from what was estimated.

It has, however, the following advantages over other possible choices:

- The LEI index has been more successful at predicting economic recoveries than at forecasting recessions.
- The Conference Board periodically refines this measure to improve its predictive performance.
- The LEI index offers investors and analysts a best guess (based on underlying data) of what the economy may do in the next six to nine months.

We define the factor λ_{fin} by means of the LEI index as follows: On the interval $(T_{i-1}, T_i]$, let $\Delta(\%LEI)$ denote the reported percent change of LEI index on the month prior to the payment date T_{i-1} . Then define

$$\lambda_{fin}(t; i) = \begin{cases} 1 & \text{if } \Delta(\%LEI) \text{ is positive,} \\ 1 - \frac{|\Delta(\%LEI)|}{100} & \text{if } \Delta(\%LEI) \text{ is negative and } |\Delta(\%LEI)| < 1, \\ 0 & \text{if } \Delta(\%LEI) \text{ is negative and } |\Delta(\%LEI)| \geq 1, \end{cases} \quad (3.3)$$

where $1 \leq i \leq N_m$. This factor then gives a measurement on how good or bad the economy is doing. In particular, the worse the state economy is, the smaller λ_{fin} is. On the other hand, the better the economic situation is, the closer to 1 λ_{fin} is.

3.1.3 Liquidity of the Mortgage Market: $\lambda_{liq}(t)$

The second factor $\lambda_{liq}(t)$ provides a measurement of the state of the housing market. Due to the crisis of 2007, the housing market experienced difficulties due to the sharp drop in the prices of residential homes, and therefore it became very difficult to sell houses. To define $\lambda_{liq}(t)$ we will use the *Housing Market Index (HMI)*, an index that assesses the current market for new single-family home sales along with builder expectations of future trends. The HMI is published monthly by the National Association of Home Builders and Wells Fargo. It is released in the same month it reports on. Revisions to this index tend to be minor. It is available on the Internet at <http://www.nahb.org>.

The HMI is a weighted average of the results from three main questions of a monthly survey conducted by the National Association of Home Builders:

- What are the current conditions for new single-family home sales? (weight: 59%)
- What are the expectations of new single-family home sales for the next six months? (weight: 14%)
- What is the traffic of prospective home buyers at new home sites? (weight: 27%)

It has a scale from 0 to 100, where 0 means that virtually everyone agreed conditions were poor, while 100 indicates that everyone believed the conditions were good. The index is adjusted for seasonal factors.

The HMI has the following limitations:

- The statistical sample is fairly small: the survey is based on responses from 400 builders out of a total membership of 72,000.
- The data is not broken down regionally, making it hard to identify areas of the country that are experiencing strong or weak demand for new single-family homes.

It has, nevertheless, the following advantages:

- The HMI is based on responses directly from homebuilders, who have the best pulse on the current and future homebuilding trends.

- As the HMI is released on the same month it reports on, it is known before any of the other major monthly housing reports are out.
- This index has a proven track record of being a decent leading indicator of future home sales.

We define the factor λ_{fin} by means of the HMI as follows: On the interval $(T_{i-1}, T_i]$, let (HMI) denote the index reported on the month prior to the payment date T_{i-1} . Then define

$$\lambda_{liq}(t; i) = \frac{(\text{HMI})}{100}, \quad (3.4)$$

for $1 \leq i \leq N_m$.

There is also another possible choice for λ_{fin} , as it may be defined with the help of the *Market Composite Index from the Weekly Mortgage Applications Survey by the Mortgage Bankers Association of America*. This is the best indicator of total mortgage application activity. It tracks all mortgage applications during the latest week, regardless of whether they were to purchase or refinance a home. This index covers conventional and government-backed mortgage applications, as well as major types of mortgage maturities: 30-year fixed, 15-year fixed, and adjustable rate mortgages. We use the definition given by (3.4), as the information about the HMI is easier to obtain.

3.1.4 Quality of the Mortgage: $\lambda_{cr}(t)$

The third and final discount factor λ_{cr} is assumed to be a constant for the entire life of the loan, as it is a measure of how trustworthy a borrower is, and this is typically determined at the origination of the mortgage. The probability of default is the likelihood that the loan will not be repaid and will fall into default. It is calculated for each borrower a group of borrowers with similar attributes. The credit score of the borrower is taken into account when calculating this probability. The simplest approach to find this score, taken by many banks and other lending institutions, is to use external ratings agencies. For the credit score, mortgage lenders usually use the *FICO score*, which is usually intended to show the likelihood that a borrower will default on a loan.

The FICO score was developed by the Fair Isaac Corporation. It is today's most commonly used scoring system. FICO scores range from 300-850 (higher FICO scores are better). Lenders buy FICO scores from three national credit reporting agencies (also called credit bureaus): Equifax, Experian and TransUnion.

At origination ($t = 0$) define $\lambda_{cr}(0)$ to be equal to the probability of default of a borrower as evidenced by his credit score. Then, on the interval $(T_{i-1}, T_i]$, we let all the future values to be same as this, that is to say:

$$\lambda_{cr}(t; i) = \lambda_{cr}(0), \quad (3.5)$$

where $1 \leq i \leq N_m$.

See the Consumer Federation of America and Fair Isaac Corporation web page, <http://www.pueblo.gsa.gov>, to learn more details on the FICO scoring process and the credit rating of borrowers.

3.2 Coherent Boundary Conditions with Stochastic Terms

Mortgage values calculated by the standard model from chapter 2 with the modifications given by expressions (3.1), (3.2), (3.3), (3.4), and (3.5), are closer to real world mortgage values calculated with in current economic crisis environment. Nevertheless, there is still a need for better values, especially when the house price h and the interest rate r approach extreme values such as 0 or ∞ . When the state variables experience these kind limiting conditions, there seem to be random behavior from both lenders and borrowers and this results in inaccurate mortgage values if we do not take additional information into account. To account for such unpredictable behavior we introduce in this section stochastic boundary conditions, where the value of the mortgage on certain portion of the boundary is equal to a random variable.

Before we move on to define the stochastic boundary conditions, recall that we remarked in 2.2.2 that the choice of boundary conditions (2.18), (2.19), (2.20)-(2.22), (2.23)-(2.25), (2.26)-(2.28) and (2.29)-(2.31) is not consistent across the literature. This may seem as just a simple disagreement in the mathematical finance community, but it turns out that in most of the previous work, such as [14], [17], [15], and [25], some of the boundary conditions are taken

to be the restriction of the valuation PDE (2.60) to the boundary. This approach is hard to justify however, as a second order PDE such as (2.60), would lead to second order boundary conditions. Also, there is no general mathematical theory available and hence there is no good interpretation for a boundary condition that comes from the restriction of the PDE to the boundary. It is hence difficult to validate this approach and we pay special attention in this thesis to define conditions that are sound from the perspective of both mathematics and finance.

Now, remember that we saw in equations (2.9) and (2.13) that the temporal or payment date conditions for the value of the mortgage V are given by:

$$\begin{aligned} V(T_{N_m}; N_m) &= \min\{MP, h\}, \\ V(T_i; i) &= \min\{V(T_i+; i+1) + MP, h\}, \end{aligned}$$

where $1 \leq k \leq N_m - 1$. Also from equations (2.22), (2.25), (2.28), and (2.31), the standard boundary conditions for this function are:

- at $h = 0$, $V = 0$,
- as $h \rightarrow \infty$, $\frac{\partial V}{\partial h} \rightarrow 0$,
- at $r = 0$, $V(h, 0, t) = A(r, t)$,
- as $r \rightarrow \infty$, $V = 0$.

In addition, the free boundary condition was defined as inequality (2.32):

$$V(h, r, t) \leq TD(t), \quad \forall t,$$

where remember that prepayment first occurs when $V(h, r, t) = TD(t)$.

The boundary conditions that we consider in this thesis are a departure from the standard condition shown above. In particular, our conditions are coherent with the financial reasoning used by previous literature and, moreover, they are also consistent from the mathematical point of view. We defined as follows:

- As $h \rightarrow \infty$,

$$V = \Phi(r, t), \quad (3.6)$$

where Φ is a function that has the property that $\Phi \rightarrow 0$ as $r \rightarrow \infty$.

- At $r = 0$,

$$V = \min\{h, TD(t)\}. \quad (3.7)$$

- At $r \rightarrow \infty$,

$$V = 0. \quad (3.8)$$

The main departure from the standard conditions used by the literature is that we introduce define a stochastic boundary condition when h is 0, and hence we set

$$V = Y(\omega). \quad (3.9)$$

The function $Y(\omega)$ is a random function defined on a very special probability space that will be explained in detail in the next chapter (see section 4.1). It represents the source of randomness when the house price becomes 0 (or in the real world, approaches 0) and it accounts for the rather irrational behavior of the housing market under such situation. As for the function Φ in (3.9), we actually choose it to satisfy the requirement that V must be sufficiently smooth to guarantee a solution to PDE (2.60).

The mortgage value V can now be calculated as a solution to the valuation PDE (2.60), that satisfies the boundary conditions given by (3.6)-(3.9). Nevertheless, when a random variable appears as part of the boundary conditions, then the value of mortgage is also dependent on the probability space of choice, and hence we have that $V = V(h, r, t, \omega)$. Similarly, equation (2.60) is not deterministic anymore, but it is a stochastic PDE. We will discuss how to solve this stochastic equation in chapter 5.

CHAPTER 4. REVIEW ON THE WIENER-ITO CHAOS EXPANSIONS

In this chapter we give a quick overview of the Wiener-Ito chaos expansion.

4.1 The White Noise Probability Space

The random function $Y(\omega)$ introduced in the last chapter in (3.9) is defined on an abstract probability space. This space can be chosen in several different way, but we will chose it to be the *white noise probability space*, which very special and useful from the perspective of numerical analysis, as it provides the foundation for the theoretical solution of stochastic PDEs and the numerical schemes that approximate this theoretical solutions. We start first with a few definitions and facts, see [12] and [23] for more details.

Let $\mathcal{S} = \mathcal{S}(\mathbb{R})$ be the *Schwartz space* of rapidly decreasing smooth functions on \mathbb{R} with the usual topology, this means that $\phi_m \rightarrow \phi$ in \mathcal{S} if $\lim_{m \rightarrow \infty} \|x^\alpha D^\beta(\phi_m - \phi)\|_\infty = 0$, for any pair of multi-indices α, β (see the next section below for an explanation of the multi-index notation) and let $\mathcal{S}' = \mathcal{S}'(\mathbb{R})$ be the dual space of \mathcal{S} , which is called the space of *tempered distributions*. Let \mathcal{B} denote the family of all Borel subsets of \mathcal{S}' equipped with the weak-star topology, where $T_m \rightarrow T$ in \mathcal{S}' if $T_m(\phi) \rightarrow T(\phi)$ for all $\phi \in \mathcal{S}$. If $\omega \in \mathcal{S}'$ and $\phi \in \mathcal{S}$, we denote the action of the tempered distribution ω on the test function ϕ by

$$\omega(\phi) = \langle \omega, \phi \rangle.$$

By the Bochner-Minlos theorem¹, there exists a probability measure μ on \mathcal{S}' such that

$$\int_{\mathcal{S}'} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2}$$

¹The Bochner-Minlos' theorem broadly states that "a cylindrical measure on the dual of a nuclear space is a Radon measure if its Fourier transform is continuous." It thus guarantees the existence of the measure μ . See [12] for a proof of this fact.

for all $\phi \in \mathcal{S}$, and where we use the notation $\|\phi\| = \|\phi\|_{L^2(\mathbb{R})}$. This measure μ is called the *white noise probability measure* and the measure space $(\mathcal{S}', \mathcal{B}, \mu)$ is called the *white noise probability space*.

Next, we define the *smoothed white noise process* as a map $w : \mathcal{S} \times \mathcal{S}' \longrightarrow \mathbb{R}$ which is given by

$$w(\phi, \omega) = \langle \omega, \phi \rangle,$$

for all $\phi \in \mathcal{S}$ and all $\omega \in \mathcal{S}'$. From this definition we can construct a Wiener process W_t that will allow us to define a stochastic integral, which in turn will help us define the components of the Wiener-Ito chaos expansion.

We outline the construction of this Wiener process as the following set of steps:

1. First we use the *Ito isometry*, which in this context can be written as

$$E[w(\phi, \cdot)^2] = \|\phi\|^2,$$

where

$$E[w(\phi, \cdot)^2] = \int_{\mathcal{S}'} w(\phi, \omega)^2 d\mu(\omega) = \int_{\mathcal{S}'} \langle \omega, \phi \rangle^2 d\mu(\omega).$$

2. Second, with the Ito isometry as before, define, for arbitrary $\psi \in L^2(\mathbb{R})$

$$\langle \omega, \psi \rangle = \lim \langle \omega, \phi_n \rangle,$$

where $\phi_n \in \mathcal{S}$ and $\phi_n \rightarrow \psi$ in $L^2(\mathbb{R})$.

3. Then, we use the previous step, define:

$$\tilde{W}_t(\omega) = \langle \omega, \chi_{[0,t]} \rangle$$

for all $t \geq 0$.

4. Finally, it can be proved that \tilde{W}_t has a continuous modification W_t , i.e. $\tilde{W}_t = W_t$, a.s. and that W_t can be extended for all $t \in \mathbb{R}$. This continuous process W_t is a standard Wiener process.

See [23] for more details.

From the construction of the standard Wiener process W_t , we establish a relation between w and W_t is given by

$$w(\phi, \omega) = \int_{\mathbb{R}} \phi(t) dW_t(\omega),$$

where the integral on the right is an Ito integral (for a precise definition of these kind of integral, see [24]).

4.2 Wiener Ito Chaos Expansion

This section begins with a few definitions that are needed in order to derive a very important fact, theorem 4.2.1 below. This theorem forms the foundation of the numerical method that will be used in 5 to find an approximate solution to the mortgage valuation of this thesis.

We start with the *multi-index notation*, which is widely used in the context of Wiener-Ito chaos series expansions. Denote the set of all finite *multi-indices* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, where $\alpha_i \in \mathbb{N}$ and $m = 1, 2, 3, \dots$, by \mathcal{J} . The *order* of a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ is defined as $|\alpha| = \sum_{i=1}^m \alpha_i$. Addition of multi-indices is defined entrwise as $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_m + \beta_m)$, while the factorial of a multi-index is defined as $\alpha! = \alpha_1! \alpha_2! \dots \alpha_m! = \prod_{j=1}^m \alpha_j!$.

Now, let the *Hermite polynomials* $p_n(x)$ be defined by

$$p_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} \left(e^{-x^2/2} \right), \quad n = 0, 1, 2, \dots \quad (4.1)$$

As an illustration, the first few Hermite polynomials are $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2 - 1$, $p_3(x) = x^3 - 3x$, $p_4(x) = x^4 - 6x^2 + 3$, $p_5(x) = x^5 - 10x^3 + 15x$, etc.

Next, let e_k be the k -th *Hermite function*, which defined with the aid of the polynomials given by (4.1) and given by

$$e_k(x) = \frac{p_{k-1}(\sqrt{2}x)}{\pi^{1/4}((k-1)!)^{1/2}e^{x^2/2}}, \quad k = 1, 2, 3, \dots \quad (4.2)$$

It can be proved that the set of Hermite functions $\{e_k\}_{k \geq 1}$ constitutes an orthonormal basis for the Hilbert space $L^2(\mathbb{R})$. Moreover, $e_k \in \mathcal{S}$ for all k . See [23] for more details.

From the Hermite functions defined in (4.2), we now define the following random variables:

$$\theta_k(\omega) = \langle \omega, e_k \rangle = \int_{\mathbb{R}} e_k(x) dW_t(\omega). \quad (4.3)$$

Finally, we combine the Hermite polynomials (4.1) and the random variables (4.3) as

$$P_\alpha(\omega) = \prod_{j=1}^m p_{\alpha_j}(\theta_j), \quad (4.4)$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathcal{J}$. For example, if $\alpha = \epsilon_k = (0, 0, \dots, 1)$, that is α is the multi-index that has a 1 in the k -th entry, but 0 elsewhere, and $\beta = (3, 0, 2)$, then

$$\begin{aligned} P_\alpha(\omega) &= p_1(\theta_k) = \langle \omega, e_k \rangle, \\ P_\beta(\omega) &= p_3(\theta_1) \cdot p_0(\theta_2) \cdot p_2(\theta_3) = (\theta_1^3 - 3\theta_1) \cdot 1 \cdot (\theta_3^2 - 1). \end{aligned}$$

It turns out that the family $\{P_\alpha\}_{\alpha \in \mathcal{J}}$ forms an orthogonal basis for the L^2 -space that is defined on the white noise probability space:

$$L^2(\mu) = L^2(\mathcal{S}', \mathcal{B}, \mu) = \left\{ X : \mathcal{S}' \longrightarrow \mathbb{R} \mid \|X\|_{L^2(\mu)}^2 = \int_{\mathcal{S}'} X^2(\omega) d\mu(\omega) < \infty \right\}. \quad (4.5)$$

We summarize this fact as a theorem, the *Wiener-Ito chaos expansion theorem*.

Theorem 4.2.1. *For all $X \in L^2(\mu)$, there exist uniquely determined numbers $X_\alpha \in \mathbb{R}$ such that*

$$X(\omega) = \sum_{\alpha} X_\alpha P_\alpha(\omega). \quad (4.6)$$

Moreover,

$$\|X\|_{L^2(\mu)}^2 = \sum_{\alpha} \alpha! X_\alpha^2. \quad (4.7)$$

Theorem 4.2.1 constitutes the main tool to develop numerical methods for stochastic PDEs. See [12] for more.

We remarked in section 3.2 that when considering stochastic boundary conditions, such as (3.9), the valuation equation (2.60) becomes a stochastic PDE, since now the mortgage value V also depends on the random element ω . The modeling approach of this thesis takes the white noise probability space $(\mathcal{S}', \mathcal{B}, \mu)$ as the domain of the random variable Y that appears in (3.9). This choice is made for the sake of the numerical simulations, as we wish to approximate the solution to (2.60) with the help of the chaos expansion series (4.6), as we will describe in 5.1.

4.3 Stochastic Distributions

In general, the solution to a stochastic PDE is not a function but a kind of distribution, which we briefly define here for completeness of the discussion. See [12] and [23] for more details.

Analogous to the deterministic test functions, given as elements of the Schwartz space $\mathcal{S}(\mathbb{R})$, and the tempered distributions, defines to be the elements of $\mathcal{S}'(\mathbb{R})$, there is also a space of *stochastic test functions*, denoted by (\mathcal{S}) , and a space of *stochastic distributions*, denoted by $(\mathcal{S})^*$. The space of stochastic test functions is called the *Hida test function space*, while the space of stochastic distributions is called the *Hida distribution space*. These two spaces are defined on the white noise probability space.

The stochastic test functions are defined as follows: We say that a sum $f = \sum_{\alpha} f_{\alpha} P_{\alpha} \in L^2(\mu)$ belongs to the space (\mathcal{S}) if

$$\sum_{\alpha \in \mathcal{J}} \alpha! f_{\alpha}^2 \left\{ \prod_{j=1}^{\infty} (2j)^{\alpha_j} \right\}^k < \infty. \quad (4.8)$$

for all $k < \infty$. On the other hand, the stochastic distributions are defined as follows: We say that a formal sum $F = \sum_{\alpha} F_{\alpha} P_{\alpha}$ belongs to the space $(\mathcal{S})^*$ if there exists $q < \infty$ such that

$$\sum_{\alpha \in \mathcal{J}} \alpha! F_{\alpha}^2 \left\{ \prod_{j=1}^{\infty} (2j)^{\alpha_j} \right\}^{-q} < \infty. \quad (4.9)$$

Expressions (4.8) and (4.9) are growth conditions analogous to the growth conditions that characterize Schwartz functions and tempered distributions.

It turns out that $(\mathcal{S})^*$ is the dual space of (\mathcal{S}) , where the action of a stochastic distribution $F = \sum_{\alpha} F_{\alpha} P_{\alpha} \in (\mathcal{S})^*$ on a stochastic test function $f = \sum_{\alpha} f_{\alpha} P_{\alpha} \in (\mathcal{S})$ is given by

$$\langle F, f \rangle = \sum_{\alpha} \alpha! f_{\alpha} F_{\alpha}.$$

In addition, we remark that $(\mathcal{S}) \subset L^2(\mu) \subset (\mathcal{S})^*$ and that these inclusions are strict.

The solution to the mortgage valuation model of this thesis is, in principle, a stochastic distribution belonging to $(\mathcal{S})^*$. Nevertheless, in the very special case of the valuation PDE (2.60) where the random terms do not appear as factors, the situation is much more simpler

and we do not have to be concerned with stochastic distributions. See [12] for more information on stochastic PDEs with multiplicative random terms and the numerical methods involved.

CHAPTER 5. SIMULATION OF THE MODEL

In this chapter we analyze the simulation of our model and present some numerical results.

5.1 Chaos Expansions Applied to the Model

We now turn to discuss and explain the solution method we will use to find a solution of the valuation PDE given by equation (2.60). As we mentioned in section 3.2, when we considered the coherent conditions with stochastic terms given by (3.6)-(3.9), equation (2.60) becomes a stochastic PDE, and in principle we must find $V(h, r, t, \omega)$ for all possible h, r, t , where $0 \leq h < \infty$, $0 \leq r < \infty$, $T_{i-1} < t \leq T_i$ for $i = 1, \dots, N_m$, and $\omega \in (\mathcal{S}', \mathcal{B}, \mu)$. Ideally we would like to obtain an explicit solution formula for the mortgage value V , but in reality such formula is rather impossible to find. The best we can get is a series representation for V involving the Wiener-Ito chaos expansion (4.6), where the deterministic coefficients are determined as solution of the deterministic PDE (2.60).

In order to have a well-defined and convergent series solution for V , we will assume that the random function Y that appeared as part of the boundary condition (3.9) belongs to the space $L^2(\mu)$ defined in (4.5). This technical assumption will guarantee that $V(h, r, t, \cdot) \in L^2(\mu)$ as well, see [12] for a proof of this fact. Thus, we can use theorem 4.2.1 and expand both Y and $V(h, r, t, \cdot)$ as chaos expansion series to then obtain the following two representations:

$$V(h, r, t, \omega) = \sum_{\alpha} V_{\alpha}(h, r, t) P_{\alpha}(\omega), \quad (5.1)$$

$$Y(\omega) = \sum_{\alpha} Y_{\alpha}(r, t) P_{\alpha}(\omega). \quad (5.2)$$

Notice that, even though it does not seem to be evident in (3.9), the random function Y actually has a dependence on the interest rate r and the time t , and this is reflected in the coefficients

Y_α of the chaos expansion series (5.2). Also observe that the series for V has coefficients V_α that depend on h , r , and t .

In both series, the random element ω from the white noise probability space is the variable for the orthogonal functions P_α . This is, in effect, a separation of the determinist variables h , r and t from the stochastic variable ω and we can think of 5.1 as an analogue to the separation of variables method used to solve some PDEs like the heat equation. Indeed, this is one of the main features of the Wiener-Ito chaos expansion, it separates and isolates the randomness of (3.9) and passes it on to the functions P_α while the coefficients $V_\alpha(h, r, t)$ inherit the deterministic variables. Therefore, to completely find V all that remains is to discover a method to find this coefficients.

5.1.1 Sequence of Deterministic PDEs

Following a similar reasoning as the one used in the separation of variables method for PDEs, we first plug the chaos expansion series for V given by e(5.1) into the valuation PDE (2.60). This process produces a system of deterministic PDEs for the coefficients $V_\alpha(h, r, t)$, which are essentially the same as (2.60):

$$\begin{aligned} \frac{1}{2}h^2\sigma_h^2\frac{\partial^2 V_\alpha}{\partial h^2} + \rho h\sqrt{r}\sigma_h\sigma_r\frac{\partial^2 V_\alpha}{\partial h\partial r} + \frac{1}{2}r\sigma_r^2\frac{\partial^2 V_\alpha}{\partial r^2} + \kappa(\theta - r)\frac{\partial V_\alpha}{\partial r} \\ + (r - \delta)h\frac{\partial V_\alpha}{\partial h} + \frac{\partial V_\alpha}{\partial t} - rV_\alpha = 0. \end{aligned} \quad (5.3)$$

Next, we plug the chaos expansion series for Y given by (5.2) as well as the series (5.2) into the boundary condition (3.9), then set the coefficients of the resulting two series equal to each other. This yields the following boundary conditions for the functions $V_\alpha(h, r, t)$:

- At $h = 0$,

$$V_\alpha = Y_\alpha. \quad (5.4)$$

- As $h \rightarrow \infty$,

$$V_\alpha = \Phi(r, t), \quad (5.5)$$

where $\Phi \rightarrow 0$ as $r \rightarrow \infty$.

- At $r = 0$,

$$V_\alpha = \min\{h, TD(t)\}. \quad (5.6)$$

- At $r \rightarrow \infty$,

$$V_\alpha = 0. \quad (5.7)$$

These coefficients also satisfy the temporal conditions given by (2.9) and (2.13):

$$V_\alpha(T_{N_m}; N_m) = \min\{MP, h\}, \quad (5.8)$$

$$V_\alpha(T_i; i) = \min\{V_\alpha(T_i+; i+1) + MP, h\}. \quad (5.9)$$

The mathematical justification for the steps taken to derive the PDE (5.3), the boundary conditions (5.4)-(5.7), and the temporal conditions (5.8) and (5.9) lies on the fact that the Wiener-Ito chaos expansion is defined on the Hilbert space $L^2(\mu)$. For more details and proofs see [12].

The free boundary condition (2.32) is a special case and it is inherited only by the first coefficient of the chaos expansion series (5.1), when $\alpha = 0$. Hence, we have that

$$V_0(h, r, t) \leq TD(t), \quad \forall t. \quad (5.10)$$

Now, remember that the functions P_α in (4.4) are orthogonal in the space $L^2(\mu)$ defined in (4.5), this implies that for any Wiener-Ito chaos expansion of a random function X , the first coefficient is actually the expectation of X , namely $X_0 = E[X]$. Hence, applying this fact to series (5.1) we have that $V_0 = E[V(h, r, t, \omega)]$ and thus obtain the following alternative version of (5.10):

$$E[V(h, r, t, \omega)] \leq TD(t), \quad \forall t.$$

Provided we find all of the coefficients V_α , the chaos expansion series (5.1) provides an explicit formula for the value of the mortgage V . In reality, however, we will be satisfied with less and will just concentrate on finding a good enough number of coefficients in order to compute a few statistical moments. In particular, the expectation $E[V]$ and the variance $\text{Var}[V]$ are two important moments for most for most the participants of the housing market, especially

investors. Therefore, we will solve the system of PDEs given by (5.3) just for sufficiently large $|\alpha| := \sum_{j=1}^m \alpha_j$, the choice depending on the statistical moment of interest.

Notice that the separation of the random and deterministic parts of the mortgage value V allows us to use standard numerical methods to solve the problem given by (5.3), (5.4)-(5.7), (5.8) and (5.9), and (5.10) for the deterministic coefficients $V_\alpha(h, r, t)$. The discussion about the numerical analysis of this problem is presented in the next section.

Before we move on to the next section, we remark that, in order to guarantee the well posedness of equation (2.60) and therefore of equations (5.3), we can assume that the random function Y belongs to a more general space, namely a Sobolev space, rather than just the space $L^2(\mu)$. We can thus impose the condition that $Y \in \mathcal{L}^2(\Omega, H^{1/2}(\Omega))$, where $\Omega = (\mathcal{S}', \mathcal{B}, \mu)$. Nevertheless, after some empirical investigation regarding some possible choices for Y , we believe that there is no need to worry about unstability of the involved PDE and we leave this technical requirement out of our numerical experiments.

5.2 Discretization of the Coefficient PDEs

Since equations (5.3) are of parabolic type, it is natural to apply finite difference methods to discretize them. This is the approach we will take in this thesis, and in particular we will use the *Crank-Nicolson scheme* to find the numerical solutions. The Crank-Nicolson scheme is a well-known numerical method that works particularly well with PDEs like (5.3). Refer to [27] for details regarding the stability, consistency, and convergence of this method.

5.2.1 Discretization of the Domain

First we will start with a short description of the the discretization of the domain of the coefficients V_α . Recall that this domain is given by three kind of intervals and we can described as $[0, \infty) \times [0, \infty) \times (T_{i-1}, T_i]$, where we have that $1 \leq i \leq N_m$. To simplify the involved manipulations, we will work with the reverse time τ given by (2.33) in which case the domain is described as $[0, \infty) \times [0, \infty) \times [\tau_k, \tau_{k+1})$, where we have that $1 \leq k \leq N_m$. Observe that this is, in fact, a collection of domains, one for each month in between payment times. Now, to perform

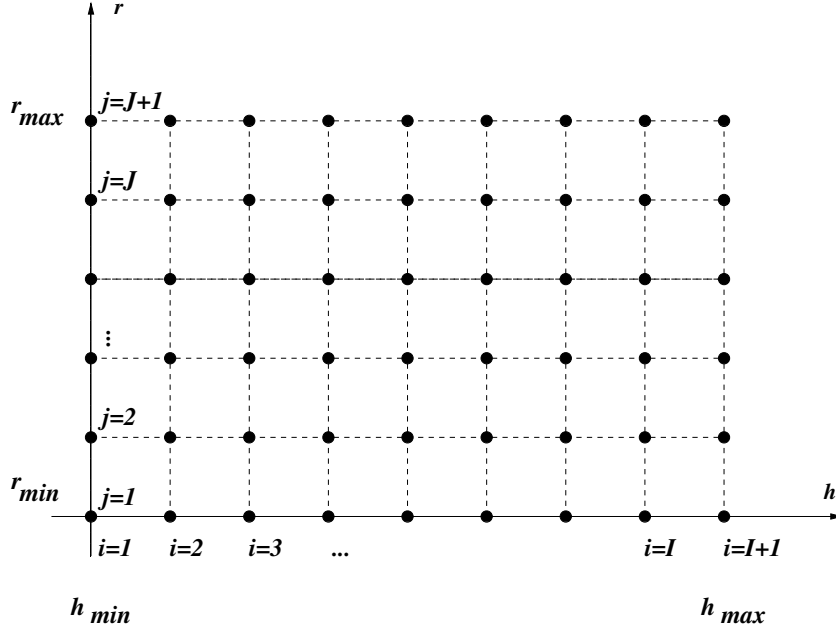


Figure 5.1 Discretization of the domain

the necessary numerical simulations, we need to change this unbounded infinite domain into a bounded finite one, we thus change it to be $[h_{min}, h_{max}] \times [r_{min}, r_{max}] \times [\tau_k, \tau_{k+1})$, where $h_{min} = 0$ and $r_{min} = 0$, while h_{max} and r_{max} are large enough constants.

We remark that with the reverse time, (5.3) changes and essentially becomes the same as (2.61).

The Crank-Nicolson scheme is a method that finds a numerical approximation by time steps, that is to say, for each fixed time, we will solve system of algebraic equations whose solution will help us solve the next time step equations. We present in figure 5.1 a sketch of the discretization of the spacial domain $[h_{min}, h_{max}] \times [r_{min}, r_{max}]$ for an arbitrary fixed time step.

The discretization of the h -axis is given by

$$\Delta h = \frac{h_{max} - h_{min}}{I},$$

$$h_i = h_{min} + (i - 1)\Delta h, \quad i = 1, \dots, I + 1,$$

where I denotes the number of desired intervals in the h -direction. Next, the discretization of

the r -axis is given by

$$\Delta r = \frac{r_{max} - r_{min}}{J},$$

$$r_j = r_{min} + (j - 1)\Delta r, \quad j = 1, \dots, J + 1,$$

where J denotes the number of desired intervals in the r -direction. Finally, the discretization of the reverse time τ is given by

$$\Delta \tau = \frac{\tau_{k+1} - \tau_k}{N},$$

$$\tau_n = \tau_k + (n - 1)\Delta \tau, \quad n = 1, \dots, N,$$

where N denotes the number of desired intervals in the τ -direction.

The discretization chosen here is standard, see [27] for more information.

5.2.2 Numerical Scheme

The numerical method we use is the Crank-Nicolson scheme, which we can briefly describe as the average of the forward Euler scheme and the backward Euler scheme in time. This method is based on the approximation of the partial derivatives in space by central differences and the approximation of the partial derivative in time by the trapezoidal finite difference rule. These finite difference rules make use of three points from each the discretization of the space variables and two points from the discretization in time. Figure 5.2 presents the typical three-point stencil for the Crank-Nicolson scheme using one variable.

In order to simplify the rest of the presentation, we will denote the generic coefficient function V_α by F , and we will also use the following notation throughout the rest of this chapter:

$$F_{i,j}^n = F(h_i, r_j, t_n),$$

$$D_h^+ F_{i,j}^n = \frac{F_{i+1,j}^n - F_{i,j}^n}{\Delta h},$$

$$D_h^- F_{i,j}^n = \frac{F_{i,j}^n - F_{i-1,j}^n}{\Delta h},$$

$$D_h^0 F_{i,j}^n = \frac{F_{i+1,j}^n - F_{i-1,j}^n}{2\Delta h},$$

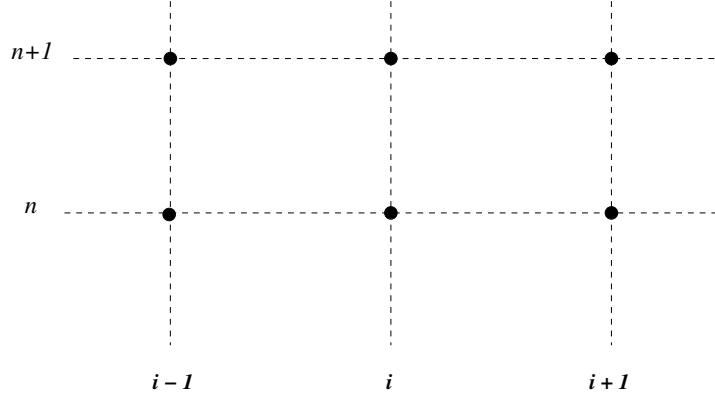


Figure 5.2 Typical Crank-Nicolson stencil

and etc., with analogous definitions for the variables r and τ .

We can then list the different Crank-Nicolson derivative approximations for all the different kinds of partial derivatives that occur in equation (5.3) with the reverse time τ :

- First order partial in τ :

$$\frac{\partial F}{\partial \tau} \approx D_{\tau}^{+} F_{i,j}^n = \frac{F_{i,j}^{n+1} - F_{i,j}^n}{\Delta t}. \quad (5.11)$$

- First order partial in h :

$$\frac{\partial F}{\partial h} \approx \frac{1}{2} \left[D_h^0 F_{i,j}^{n+1} + D_h^0 F_{i,j}^n \right] = \frac{1}{2} \left[\frac{F_{i+1,j}^{n+1} - F_{i-1,j}^{n+1}}{2\Delta h} + \frac{F_{i+1,j}^n - F_{i-1,j}^n}{2\Delta h} \right]. \quad (5.12)$$

- First order partial in r :

$$\frac{\partial F}{\partial r} \approx \frac{1}{2} \left[D_r^0 F_{i,j}^{n+1} + D_r^0 F_{i,j}^n \right] = \frac{1}{2} \left[\frac{F_{i,j+1}^{n+1} - F_{i,j-1}^{n+1}}{2\Delta r} + \frac{F_{i,j+1}^n - F_{i,j-1}^n}{2\Delta r} \right]. \quad (5.13)$$

- Second order partial in h :

$$\begin{aligned} \frac{\partial^2 F}{\partial h^2} &\approx \frac{1}{2} \left[D_h^{+} D_h^{-} F_{i,j}^{n+1} + D_h^{+} D_h^{-} F_{i,j}^n \right] \\ &= \frac{1}{2} \left[\frac{F_{i+1,j}^{n+1} - 2F_{i,j}^{n+1} + F_{i-1,j}^{n+1}}{(\Delta h)^2} + \frac{F_{i+1,j}^n - 2F_{i,j}^n + F_{i-1,j}^n}{(\Delta h)^2} \right]. \end{aligned} \quad (5.14)$$

- Second order partial in r :

$$\begin{aligned} \frac{\partial^2 F}{\partial r^2} &\approx \frac{1}{2} \left[D_r^{+} D_r^{-} F_{i,j}^{n+1} + D_r^{+} D_r^{-} F_{i,j}^n \right] \\ &= \frac{1}{2} \left[\frac{F_{i,j+1}^{n+1} - 2F_{i,j}^{n+1} + F_{i,j-1}^{n+1}}{(\Delta r)^2} + \frac{F_{i,j+1}^n - 2F_{i,j}^n + F_{i,j-1}^n}{(\Delta r)^2} \right]. \end{aligned} \quad (5.15)$$

- Second order crossed partial:

$$\begin{aligned}
\frac{\partial^2 F}{\partial h \partial r} &\approx \frac{1}{2} \left[D_h^0 D_r^0 F_{i,j}^{n+1} + D_h^0 D_r^0 F_{i,j}^n \right] \\
&= \frac{1}{2} \left[\frac{F_{i+1,j+1}^{n+1} - F_{i+1,j-1}^{n+1} - F_{i-1,j+1}^{n+1} + F_{i-1,j-1}^{n+1}}{4\Delta h \Delta r} \right. \\
&\quad \left. + \frac{F_{i+1,j+1}^n - F_{i+1,j-1}^n - F_{i-1,j+1}^n + F_{i-1,j-1}^n}{4\Delta h \Delta r} \right]. \tag{5.16}
\end{aligned}$$

- In addition to the derivative approximations, we also have:

$$F \approx \frac{1}{2} \left[F_{i,j}^{n+1} + F_{i,j}^n \right]. \tag{5.17}$$

With these approximations to the numerical scheme will have an order of accuracy of second order in h , r , and t , that is to say $O(\Delta h^2 + \Delta r^2 + \Delta \tau^2)$. See [27] for more a proof of this fact.

The numerical scheme is obtained by replacing the partial derivatives that appear in (5.3) (with the reverse time τ) by the corresponding expressions from (5.11)-(5.11). This yields

$$\begin{aligned}
\frac{F_{i,j}^{n+1} - F_{i,j}^n}{\Delta \tau} &= \frac{1}{2} h_i^2 \sigma_h^2 \left[\frac{F_{i+1,j}^{n+1} - 2F_{i,j}^{n+1} + F_{i-1,j}^{n+1}}{2(\Delta h)^2} + \frac{F_{i+1,j}^n - 2F_{i,j}^n + F_{i-1,j}^n}{2(\Delta h)^2} \right] \\
&\quad + \rho \sigma_h \sigma_r h_i \sqrt{r_j} \left[\frac{F_{i+1,j+1}^{n+1} - F_{i+1,j-1}^{n+1} - F_{i-1,j+1}^{n+1} + F_{i-1,j-1}^{n+1}}{8\Delta h \Delta r} \right. \\
&\quad \left. + \frac{F_{i+1,j+1}^n - F_{i+1,j-1}^n - F_{i-1,j+1}^n + F_{i-1,j-1}^n}{8\Delta h \Delta r} \right] \\
&\quad + \frac{1}{2} r_j \sigma_r^2 \left[\frac{F_{i,j+1}^{n+1} - 2F_{i,j}^{n+1} + F_{i,j-1}^{n+1}}{2(\Delta r)^2} + \frac{F_{i,j+1}^n - 2F_{i,j}^n + F_{i,j-1}^n}{2(\Delta r)^2} \right] \\
&\quad + \kappa(\theta - r_j) \left[\frac{F_{i,j+1}^{n+1} - F_{i,j-1}^{n+1}}{4\Delta r} + \frac{F_{i,j+1}^n - F_{i,j-1}^n}{4\Delta r} \right] \\
&\quad + (r_j - \delta) h_i \left[\frac{F_{i+1,j}^{n+1} - F_{i-1,j}^{n+1}}{4\Delta h} + \frac{F_{i+1,j}^n - F_{i-1,j}^n}{4\Delta h} \right] \\
&\quad - \frac{r_j}{2} \left[F_{i,j}^{n+1} + F_{i,j}^n \right]. \tag{5.18}
\end{aligned}$$

We simplify this expression by multiplying both sides by $\Delta \tau$ and then collecting common

terms, then, in order to simplify, we make the following definitions:

$$\begin{aligned}
a_{i,j} &= \frac{(\Delta\tau)h_i^2\sigma_h^2}{4(\Delta h)^2} - \frac{(\Delta\tau)(r_j - \delta)h_i}{4\Delta h}, \\
b_{i,j} &= \frac{(\Delta\tau)h_i^2\sigma_h^2}{2(\Delta h)^2} + \frac{(\Delta\tau)r_j\sigma_r^2}{2(\Delta r)^2} + \frac{(\Delta\tau)r_j}{2}, \\
c_{i,j} &= \frac{(\Delta\tau)h_i^2\sigma_h^2}{4(\Delta h)^2} + \frac{(\Delta\tau)(r_j - \delta)h_i}{4\Delta h}, \\
d_{i,j} &= \frac{(\Delta\tau)r_j\sigma_r^2}{4(\Delta r)^2} - \frac{(\Delta\tau)\kappa(\theta - r_j)}{4\Delta r}, \\
e_{i,j} &= \frac{(\Delta\tau)r_j\sigma_r^2}{4(\Delta r)^2} + \frac{(\Delta\tau)\kappa(\theta - r_j)}{4\Delta r}, \\
f_{i,j} &= \frac{(\Delta\tau)\rho h_i\sqrt{r_j}\sigma_h\sigma_r}{8\Delta h\Delta r},
\end{aligned}$$

where we have that $i = 2, \dots, I$, and $j = 2, \dots, J$. With $a_{i,j}$, $b_{i,j}$, $c_{i,j}$, $d_{i,j}$, $e_{i,j}$, and $f_{i,j}$ this way, then expression (5.18) becomes:

$$\begin{aligned}
& [1 + b_{i,j}] F_{i,j}^{n+1} - c_{i,j} F_{i+1,j}^{n+1} - a_{i,j} F_{i-1,j}^{n+1} - f_{i,j} [F_{i+1,j+1}^{n+1} - F_{i+1,j-1}^{n+1} - F_{i-1,j+1}^{n+1} + F_{i-1,j-1}^{n+1}] \\
& - e_{i,j} F_{i,j+1}^{n+1} - d_{i,j} F_{i,j-1}^{n+1} \\
& = [1 - b_{i,j}] F_{i,j}^n + c_{i,j} F_{i+1,j}^n + a_{i,j} F_{i-1,j}^n + f_{i,j} [F_{i+1,j+1}^n - F_{i+1,j-1}^n - F_{i-1,j+1}^n + F_{i-1,j-1}^n] \\
& + e_{i,j} F_{i,j+1}^n + d_{i,j} F_{i,j-1}^n,
\end{aligned} \tag{5.19}$$

where we have that $i = 2, \dots, I$, $j = 2, \dots, J$ and also $n = 1, \dots, N - 1$.

Expression (5.19) is essentially the numerical scheme we are looking for, where for each n , where $n = 2, \dots, N$, we have to find the unknown approximations $F_{i,j}^n$ for all i and j , where $i = 2, \dots, I$ and $j = 2, \dots, J$. Nevertheless, before we attempt to further simplify this expression, we must make a choice as to how to number this unknowns $F_{i,j}^n$, as there is more than one possibility. The choice we make is sketched in figure 5.3, where we number along the h -direction first and the r -direction second.

With the choice shown in figure 5.3, the unknown approximations $F_{i,j}^n$ can be put together

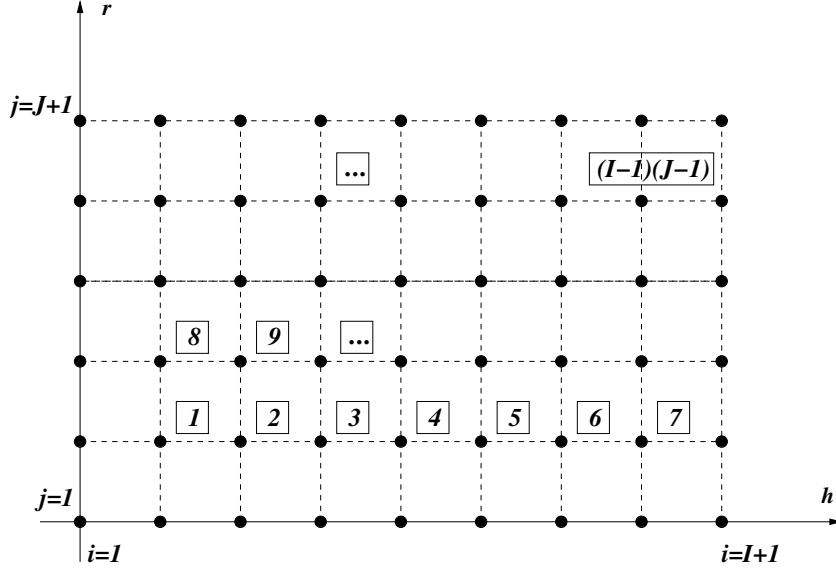


Figure 5.3 Numbering of the unknown approximations $F_{i,j}^n$

as the following vector of unknowns:

$$\vec{F}^n = \begin{pmatrix} \vec{F}_2^n \\ \vec{F}_3^n \\ \vec{F}_4^n \\ \vdots \\ \vec{F}_J^n \end{pmatrix}. \quad (5.20)$$

where we set $\vec{F}_j^n := (F_{2,j}^n, F_{3,j}^n, \dots, F_{I,j}^n)^T$, for $2 \leq j \leq J$. Observe that the vector \vec{F}^n has length $(I-1)(J-1)$. This changes expression (5.19) into the following numerical scheme:

$$\begin{aligned} & -f_{i,j}F_{i-1,j-1}^{n+1} - d_{i,j}F_{i,j-1}^{n+1} + f_{i,j}F_{i+1,j-1}^{n+1} \\ & -a_{i,j}F_{i-1,j}^{n+1} + [1 + b_{i,j}]F_{i,j}^{n+1} - c_{i,j}F_{i+1,j}^{n+1} \\ & + f_{i,j}F_{i-1,j+1}^{n+1} - e_{i,j}F_{i,j+1}^{n+1} - f_{i,j}F_{i+1,j+1}^{n+1} \\ & = \\ & f_{i,j}F_{i-1,j-1}^n + d_{i,j}F_{i,j-1}^n - f_{i,j}F_{i+1,j-1}^n \\ & + a_{i,j}F_{i-1,j}^n + [1 - b_{i,j}]F_{i,j}^n + c_{i,j}F_{i+1,j}^n \\ & - f_{i,j}F_{i-1,j+1}^n + e_{i,j}F_{i,j+1}^n + f_{i,j}F_{i+1,j+1}^n, \end{aligned} \quad (5.21)$$

where we have that $i = 2, \dots, I$, $j = 2, \dots, J$ and $n = 1, \dots, N - 1$.

The scheme presented in (5.21) is precisely the Crank-Nicolson scheme for the valuation PDE for the coefficients (5.3) with reverse time. This is an algebraic system of equations for all the approximations $F_{i,j}^n$, so our goal now is to visualize it as a matrix equation. It turns out that, due to the nature of this scheme, we obtain a coefficient matrix with banded structure. For the purpose of notational simplicity, we denote the right-hand side of (5.21) by $\text{RHS}_{i,j}^n$. Moreover, we define the following matrices that form the bands of the coefficient matrix:

- The lower matrices:

$$A_j = \begin{pmatrix} -d_{2,j} & f_{2,j} & & & 0 \\ -f_{3,j} & -d_{3,j} & f_{3,j} & & \\ & -f_{4,j} & -d_{4,j} & f_{4,j} & \\ & & \ddots & \ddots & \ddots \\ & & & -f_{I-1,j} & -d_{I-1,j} & f_{I-1,j} \\ 0 & & & & -f_{I,j} & -d_{I,j} \end{pmatrix}. \quad (5.22)$$

- The middle matrices:

$$B_j = \begin{pmatrix} 1 + b_{2,j} & -c_{2,j} & & & 0 \\ -a_{3,j} & 1 + b_{3,j} & -c_{3,j} & & \\ & -a_{4,j} & 1 + b_{4,j} & -c_{4,j} & \\ & & \ddots & \ddots & \ddots \\ & & & -a_{I-1,j} & 1 + b_{I-1,j} & -c_{I-1,j} \\ 0 & & & & -a_{I,j} & 1 + b_{I,j} \end{pmatrix}. \quad (5.23)$$

- The upper matrices:

$$C_j = \begin{pmatrix} -e_{2,j} & -f_{2,j} & & & 0 \\ f_{3,j} & -e_{3,j} & -f_{3,j} & & \\ & f_{4,j} & -e_{4,j} & -f_{4,j} & \\ & & \ddots & \ddots & \ddots \\ & & & f_{I-1,j} & -e_{I-1,j} & -f_{I-1,j} \\ 0 & & & & f_{I,j} & -e_{I,j} \end{pmatrix}. \quad (5.24)$$

In all of the above definitions, we have that $j = 2, \dots, J$. Notice that the matrices A_j , B_j , and C_j given in (5.22)-(5.24) are all of size $(I-1) \times (I-1)$.

Now, the coefficient matrix for the desired matrix system can be now written as:

$$M_F = \begin{pmatrix} B_2 & C_2 & & & 0 \\ A_3 & B_3 & C_3 & & \\ & A_4 & B_4 & C_4 & \\ & & \ddots & \ddots & \ddots \\ & & & A_{J-1} & B_{J-1} & C_{J-1} \\ 0 & & & & A_J & B_J \end{pmatrix}. \quad (5.25)$$

Observe that the matrix M_F has size $(I-1)(J-1) \times (I-1)(J-1)$.

The only missing information before we write a matrix equation is to find the right-hand-side vector that we denote by \vec{G}_F^n and write as:

$$\vec{G}_F^n = \begin{pmatrix} \vec{G}_1^n \\ \vec{G}_2^n \\ \vec{G}_3^n \\ \vdots \\ \vec{G}_{J-1}^n \end{pmatrix}, \quad (5.26)$$

where we set $\vec{G}_j^n := \left(G_{(j-1)(I-1)+1}^n, G_{(j-1)(I-1)+2}^n, \dots, G_{(j-1)(I-1)+I-1}^n \right)^T$, for $1 \leq j \leq J-1$.

Notice that the vector \vec{G}_F^n has length $(I-1)(J-1)$.

The entries of \vec{G}_F^n are known and come from different special cases of the scheme given by (5.21), as $F_{i,j}^n$ is known whenever we have $i = 1$ or $j = 1$. We now list in detail of these different special cases for completeness of the discussion:

- If $j = 2$

– and $i = 2$:

$$\begin{aligned}
& [1 + b_{i,j}] F_{i,j}^{n+1} - c_{i,j} F_{i+1,j}^{n+1} - e_{i,j} F_{i,j+1}^{n+1} - f_{i,j} F_{i+1,j+1}^{n+1} \\
& = \text{RHS}_{i,j}^n + f_{i,j} F_{i-1,j-1}^{n+1} + d_{i,j} F_{i,j-1}^{n+1} \\
& \quad - f_{i,j} F_{i+1,j-1}^{n+1} + a_{i,j} F_{i-1,j}^{n+1} - f_{i,j} F_{i-1,j+1}^{n+1} \\
& =: G_1^n.
\end{aligned}$$

– and $i = 3, 4, \dots, I - 1$:

$$\begin{aligned}
& -a_{i,j} F_{i-1,j}^{n+1} + [1 + b_{i,j}] F_{i,j}^{n+1} - c_{i,j} F_{i+1,j}^{n+1} + f_{i,j} F_{i-1,j+1}^{n+1} - e_{i,j} F_{i,j+1}^{n+1} - f_{i,j} F_{i+1,j+1}^{n+1} \\
& = \text{RHS}_{i,j}^n + f_{i,j} F_{i-1,j-1}^{n+1} + d_{i,j} F_{i,j-1}^{n+1} - f_{i,j} F_{i+1,j-1}^{n+1} \\
& =: G_l^n,
\end{aligned}$$

where $2 \leq l \leq I - 2$.

– and $i = I$:

$$\begin{aligned}
& -a_{i,j} F_{i-1,j}^{n+1} + [1 + b_{i,j}] F_{i,j}^{n+1} + f_{i,j} F_{i-1,j+1}^{n+1} - e_{i,j} F_{i,j+1}^{n+1} \\
& = \text{RHS}_{i,j}^n + f_{i,j} F_{i-1,j-1}^{n+1} + d_{i,j} F_{i,j-1}^{n+1} \\
& \quad - f_{i,j} F_{i+1,j-1}^{n+1} + c_{i,j} F_{i+1,j}^{n+1} + f_{i,j} F_{i+1,j+1}^{n+1} \\
& =: G_{I-1}^n.
\end{aligned}$$

• If $j = 3, 4, \dots, J - 1$

– and $i = 2$:

$$\begin{aligned}
& -d_{i,j} F_{i,j-1}^{n+1} + f_{i,j} F_{i+1,j-1}^{n+1} + [1 + b_{i,j}] F_{i,j}^{n+1} - c_{i,j} F_{i+1,j}^{n+1} - e_{i,j} F_{i,j+1}^{n+1} - f_{i,j} F_{i+1,j+1}^{n+1} \\
& = \text{RHS}_{i,j}^n + f_{i,j} F_{i-1,j-1}^{n+1} + a_{i,j} F_{i-1,j}^{n+1} - f_{i,j} F_{i-1,j+1}^{n+1} \\
& =: G_{k(I-1)+1}^n,
\end{aligned}$$

where $1 \leq k \leq J - 3$ (or $k = j - 2$).

– and $i = 3, 4, \dots, I - 1$:

$$\begin{aligned}
& -f_{i,j}F_{i-1,j-1}^{n+1} - d_{i,j}F_{i,j-1}^{n+1} + f_{i,j}F_{i+1,j-1}^{n+1} - a_{i,j}F_{i-1,j}^{n+1} + [1 + b_{i,j}]F_{i,j}^{n+1} - c_{i,j}F_{i+1,j}^{n+1} \\
& + f_{i,j}F_{i-1,j+1}^{n+1} - e_{i,j}F_{i,j+1}^{n+1} - f_{i,j}F_{i+1,j+1}^{n+1} \\
& = \text{RHS}_{i,j}^n \\
& =: G_{k(I-1)+l}^n,
\end{aligned}$$

where $1 \leq k \leq J - 3$ and $2 \leq l \leq I - 2$ (or $k = j - 2$ and $l = i - 1$).

– and $i = I$:

$$\begin{aligned}
& -f_{i,j}F_{i-1,j-1}^{n+1} - d_{i,j}F_{i,j-1}^{n+1} - a_{i,j}F_{i-1,j}^{n+1} + [1 + b_{i,j}]F_{i,j}^{n+1} + f_{i,j}F_{i-1,j+1}^{n+1} - e_{i,j}F_{i,j+1}^{n+1} \\
& = \text{RHS}_{i,j}^n - f_{i,j}F_{i+1,j-1}^{n+1} + c_{i,j}F_{i+1,j}^{n+1} + f_{i,j}F_{i+1,j+1}^{n+1} \\
& =: G_{k(I-1)+I-1}^n,
\end{aligned}$$

where $1 \leq k \leq J - 3$ (or $k = j - 2$).

• If $j = J$

– and $i = 2$:

$$\begin{aligned}
& -d_{i,j}F_{i,j-1}^{n+1} + f_{i,j}F_{i+1,j-1}^{n+1} + [1 + b_{i,j}]F_{i,j}^{n+1} - c_{i,j}F_{i+1,j}^{n+1} \\
& = \text{RHS}_{i,j}^n - f_{i,j}F_{i-1,j+1}^{n+1} + e_{i,j}F_{i,j+1}^{n+1} \\
& + f_{i,j}F_{i+1,j+1}^{n+1} + f_{i,j}F_{i-1,j-1}^{n+1} + a_{i,j}F_{i-1,j}^{n+1} \\
& =: G_{(J-2)(I-1)+1}^n.
\end{aligned}$$

– and $i = 3, 4, \dots, I - 1$:

$$\begin{aligned}
& -f_{i,j}F_{i-1,j-1}^{n+1} - d_{i,j}F_{i,j-1}^{n+1} + f_{i,j}F_{i+1,j-1}^{n+1} - a_{i,j}F_{i-1,j}^{n+1} + [1 + b_{i,j}]F_{i,j}^{n+1} - c_{i,j}F_{i+1,j}^{n+1} \\
& = \text{RHS}_{i,j}^n - f_{i,j}F_{i-1,j+1}^{n+1} + e_{i,j}F_{i,j+1}^{n+1} + f_{i,j}F_{i+1,j+1}^{n+1} \\
& =: G_{(J-2)(I-1)+l}^n,
\end{aligned}$$

where $2 \leq l \leq I - 2$.

– and $i = I$:

$$\begin{aligned}
& -f_{i,j}F_{i-1,j-1}^{n+1} - d_{i,j}F_{i,j-1}^{n+1} - a_{i,j}F_{i-1,j}^{n+1} + [1 + b_{i,j}]F_{i,j}^{n+1} \\
& = \text{RHS}_{i,j}^n - f_{i,j}F_{i-1,j+1}^{n+1} + e_{i,j}F_{i,j+1}^{n+1} \\
& \quad + f_{i,j}F_{i+1,j+1}^{n+1} - f_{i,j}F_{i+1,j-1}^{n+1} + c_{i,j}F_{i+1,j}^{n+1} \\
& =: G_{(J-1)(I-1)}^n.
\end{aligned}$$

From the expressions for the vector of unknowns given by (5.20), the coefficient matrix given by (5.25), and the right-hand-side vector given by (5.26), we derive the following algebraic system of equations:

$$M_F \vec{F}^{n+1} = \vec{G}_F^n, \quad (5.27)$$

where $n = 1, \dots, N - 1$.

In principle, the solution the the matrix system (5.27) is a numerical approximation to $F = V_\alpha$. Nevertheless, we did not involve the free boundary condition given by (5.10), therefore this solution is not quite correct. We will actually involve the linear complementarity formulation, as described in 1.4.3, to surmount the difficulty imposed by this free boundary condition.

5.2.3 Linear Complementarity Formulation

The free boundary condition (5.10) for the first coefficient V_0 states that $V_0(h, r, t) \leq TD(t)$, for all time t . If $F = V_0$, then we have that $F(h, r, t) \leq TD(t)$, or equivalently

$$0 \leq TD(t) - F(h, r, t). \quad (5.28)$$

Now, notice that the left and side of the valuation PDE (2.61) (and therefore of (5.3) as well), can be visualized as a linear partial differential operator acting on the function V (or the functions V_α in the case of (5.3)). We will denote the action of this operator on a function F by $\mathcal{L}\{F\}$ and will use this simplified notation for the rest of the discussion.

Using similar arguments as those that justified (1.28), we obtain the following linear com-

plementarity formulation for F :

$$\begin{aligned} (TD - F) \cdot \mathcal{L}\{C\} &= 0, \\ \mathcal{L}\{F\} &\geq 0, \\ (T - F) &\geq 0. \end{aligned} \tag{5.29}$$

Now, the numerical scheme given by (5.21) and the process that lead us to the matrix equation (5.27) can be used again to obtain the following discretized version of (5.29):

$$\begin{aligned} \left(\vec{T}D^{n+1} - \vec{F}^{n+1} \right) \cdot \left(M_F \vec{F}^{n+1} - \vec{G}_F^n \right) &= 0, \\ M_F \vec{F}^{n+1} &\geq \vec{G}_F^n, \\ \vec{T}D^{n+1} &\geq \vec{F}^{n+1}, \end{aligned} \tag{5.30}$$

where we have that $n = 1, \dots, N - 1$; \vec{F}^n , M_F , and \vec{G}_F^n are given by (5.20), (5.25), and (5.26) respectively; and finally $TD^n = TD(\tau_n)$. Observe that (5.30) is just an analogue to the constrained matrix problem given by (1.29).

We can now apply the ideas of 1.4.4 and find a solution for (5.30). The main tool to achieve this solution is the PSOR method that we describe in the next section.

5.3 PSOR Method

This section introduces the Projected Successive Over-relaxation (PSOR) method, an iterative algorithm that is widely used to find a numerical solutions to LCP problems like the one given by (5.30). This method is a generalization another iterative algorithm, the *Successive Over-relaxation (SOR) Iteration*, which generalizes *Gauss-Seidel iterations*, which in turn derives from *Jacobi iterations*. For a broader discussion regarding all these iterative methods and their application to LCPs, see [8], [10], and [18].

We start by describing a general numerical linear algebra problem regarding matrix splittings. The problem consists of finding a vector x such that $Ax = b$. Although there are many possible approaches to the solution of this matrix system, one approach would be to assume that the matrix A splits as $A = A_1 + A_2$, where A_1 is nonsingular, and also chosen such that

A_1^{-1} is “easy” to find. Then the matrix system $Ax = b$ becomes

$$(A_1 + A_2)x = b,$$

$$A_1x = b - A_2x,$$

from where we can find that

$$x = A_1^{-1}(b - A_2x).$$

This iterative formula suggests that, in order to find a solution to the original system of equations, we can define the following *fixed point iteration*:

$$x_{k+1} := A_1^{-1}(b - A_2x_k). \quad (5.31)$$

Thus, we obtain a sequence of vectors $\{x_k\}_k$ that we hope would eventually converge to the solution of the matrix system $x^* = A^{-1}b$. If the splitting $A = A_1 + A_2$ is chosen appropriately, then indeed this sequence converges and produces an *iterative solution*.

This general idea is foundation to the iterative methods that we describe next, the difference essentially lies on the choices of the matrices A_1 and A_2 .

5.3.1 Jacobi Iterations

The main idea for Jacobi iterations is a matrix splitting of the form $A = D + L + U$, where D is a diagonal matrix, L is a strict lower diagonal matrix, and U is a strict upper diagonal matrix. We then set $A_1 = D$ and $A_2 = L + U$, whence iteration (5.31) thus becomes:

$$x_{k+1} = A_1^{-1}(b - A_2x_k) = D^{-1}(b - (L + U)x_k) = D^{-1}(b - Lx_k - Ux_k).$$

If we let $x_k = (x_i^k)_{i=1}^n$, then this iteration can be formulated as

$$x_i^{k+1} = a_{ii}^{-1} \left(b_i - \sum_{j \neq i} a_{ij} x_j^k \right). \quad (5.32)$$

The sequence $\{x_k\}_k$ converges if the matrix A is strictly *diagonally dominant*. We summarize this as a theorem.

Theorem 5.3.1. *If the matrix A satisfies that*

$$0 < \sum_{j \neq i} |a_{ij}| < |a_{ii}|, \quad \text{for all } i,$$

then A is nonsingular and the Jacobi iterations defined by (5.32) converge to $x^ = A^{-1}b$.*

A proof of this theorem can be found in [18].

5.3.2 Gauss-Seidel Iterations

These iterations are a small variation of the Jacobi iterations given by (5.32), where the approximate solution is overwritten with a new value as soon as this is computed. This corresponds to setting $A_1 = D + L$ and $A_2 = U$. Then iteration (5.31) becomes:

$$\begin{aligned} x_{k+1} &= A_1^{-1}(b - A_2 x_k) \\ &= (D + L)^{-1}(b - U x_k). \end{aligned}$$

Observe that we can rewrite this since

$$\begin{aligned} x_{k+1} &= (D + L)^{-1}(b - U x_k), \\ \Rightarrow (D + L)x_{k+1} &= b - U x_k, \\ \Rightarrow D x_{k+1} &= b - L x_{k+1} - U x_k, \\ \Rightarrow x_{k+1} &= D^{-1}(b - L x_{k+1} - U x_k). \end{aligned}$$

Now, if we let $x_k = (x_i^k)_{i=1}^n$ again, then

$$x_i^{k+1} = a_{ii}^{-1} \left(b_i - \sum_{j < i} a_{ij} x_j^{k+1} - \sum_{j > i} a_{ij} x_j^k \right). \quad (5.33)$$

The $\{x_k\}_k$ defined by (5.33) converges if the matrix A is diagonally dominant, a consequence of theorem 5.3.1 above.

5.3.3 SOR Iterations

The next improvement over the Jacobi and the Gauss-Seidel iterations are the successive over-relaxation or SOR iterations, which are a modification of the Gauss-Seidel iterations

through the addition of a “relaxation” parameter, which will be denoted by ρ , in order to improve the convergence rate of the sequence of iterates. The SOR iterates are the same as the Gauss-Seidel iterates from (5.33) in the case of $\rho = 1$.

The SOR iterations derive from the following simple observation:

$$x_{k+1} = x_k + (x_{k+1} - x_k).$$

From here, the term $x_{k+1} - x_k$ is regarded as a “correction” term added to the k -th iterate x_k in order to bring it nearer to the actual solution of the matrix equation $Ax = b$, denoted again by x^* . This interpretation may speed up the convergence if we “over-correct,” especially if $x_k \rightarrow x^*$ monotonically.

Hence, the SOR iterations are defined by

$$x_{k+1} = x_k + \rho (x_{k+1}^{GS} - x_k), \quad (5.34)$$

where x_{k+1}^{GS} is the Gauss-Seidel iterate defined by (5.33). This corresponds to setting $A_1 = D + \rho L$ and $A_2 = -(1 - \rho)D + \rho U$, and therefor the iterates become:

$$\begin{aligned} x_{k+1} &= A_1^{-1}(\rho b - A_2 x_k) \\ &= (D + \rho L)^{-1}(\rho b - [-(1 - \rho)D + \rho U] x_k). \end{aligned}$$

Notice now that

$$\begin{aligned} x_{k+1} &= (D + \rho L)^{-1}(\rho b - [-(1 - \rho)D + \rho U] x_k), \\ \Rightarrow (D + \rho L)x_{k+1} &= \rho b + (1 - \rho)Dx_k - \rho Ux_k, \\ \Rightarrow Dx_{k+1} &= \rho b - \rho Lx_{k+1} - \rho Ux_k + (1 - \rho)Dx_k, \\ \Rightarrow x_{k+1} &= D^{-1}(\rho b - \rho Lx_{k+1} - \rho Ux_k) + (1 - \rho)x_k, \\ \Rightarrow x_{k+1} &= \rho D^{-1}(b - Lx_{k+1} - Ux_k) + (1 - \rho)x_k. \end{aligned}$$

Let once again $x_k = (x_i^k)_{i=1}^n$, then the SOR iteration is defined by

$$x_i^{k+1} = \rho a_{ii}^{-1} \left(b_i - \sum_{j < i} a_{ij} x_j^{k+1} - \sum_{j > i} a_{ij} x_j^k \right) + (1 - \rho) x_i^k. \quad (5.35)$$

Due to the inclusion of ρ , the SOR iterations given by (5.35) speed up the convergence of the Gauss-Seidel iterations from (5.33), nevertheless, the choice of the relaxation parameter ρ is not easy in general, as it depends on the properties of the matrix A . We do have, however a convergence result summarized in the next theorem.

Theorem 5.3.2. *If $0 < \rho < 2$ and if A is a symmetric and positive definite matrix, then the SOR iterations given by (5.35) converge to $x^* = A^{-1}b$.*

See [18] for more on the parameter ρ .

5.3.4 PSOR Iterations Applied to the Linear Complementarity Problem

We now turn to describe the projected successive over-relaxation or PSOR iterations and how this approach is used to solve the matrix LCP problem given by (5.30), which is a special case of the the general linear complementarity problem in matrix form¹:

$$\begin{aligned} (x - c) \cdot (Ax - b) &= 0, \\ Ax &\geq b, \\ x &\geq c. \end{aligned} \tag{5.36}$$

We have following convergence result:

Theorem 5.3.3. *If A is a nonsingular and positive definite matrix, then there exists a unique solution x^* of the matrix linear complementarity problem given by (5.36).*

The PSOR iterations are a variation of the SOR iterations given by (5.35), where we impose a condition in order to guarantee that the inequalities that appear in (5.36) all hold. We describe how to construct the PSOR iterates with the following algorithm:

- First choose $x_0 \geq c$.
- Then set

$$y_i^{k+1} = a_{ii}^{-1} \left(b_i - \sum_{j < i} a_{ij} x_j^{k+1} - \sum_{j > i} a_{ij} x_j^k \right). \tag{5.37}$$

¹The vector inequalities are defined entry-wise as: $y \geq z$, if $y_i \geq z_i$ for all i .

- Finally define

$$\begin{aligned} x_i^{k+1} &= \max \left\{ c_i, \rho y_i^{k+1} + (1 - \rho)x_i^k \right\} \\ &= \max \left\{ c_i, \left[y_i^{k+1} - x_i^k \right] \rho + x_i^k \right\}. \end{aligned} \quad (5.38)$$

As with the SOR iterations, if the parameter ρ satisfies that $0 < \rho < 2$, and also if we choose $x_0 \geq c$, then the PSOR iterations given by (5.37) and (5.38) converge.

We remark that the condition imposed in order to guarantee that the inequalities in the matrix LCP problem (5.36) is precisely the minimum taken in (5.38). So, the only difference between the SOR and the PSOR iterations is the test to assure that $x_i^{k+1} \geq c_i$ and hence, we can write the PSOR iterate as:

$$x_{k+1} = \max\{c, x_{k+1}^{SOR}\}, \quad (5.39)$$

where x_{k+1}^{SOR} is, in turn, the SOR iterate defined by (5.35).

As a final remark we point out that, in practice, the PSOR iterations are not continued forever, but instead, we stop when $\|x_{k+1} - x_k\| < \epsilon$, where $\epsilon > 0$ is some predetermined error tolerance.

The PSOR method is the main tool to find an approximate solution to the expected value of the mortgage value $V_0 = E[V]$, of which we present some numerical results in the next section.

5.4 Numerical Experiments

We now illustrate the practicability of the mortgage valuation model given in this thesis by performing some numerical experiments.

Remember that our approach regards the discounted mortgage value defined in (3.1) as the solution of the PDE given by (2.60), with the temporal conditions given by (2.9) and (2.13), and the coherent boundary conditions defined by (3.6)-(3.9). Moreover, the random function Y given in (3.9) is assumed to belong the space $L^2(\mu)$ defined in (4.5), which allows us to expand the mortgage value function $V(h, r, t, \omega)$ as a Wiener-Ito chaos series, as given by (5.1).

To perform the numerical simulations, we have to repeatedly solve the LCP given by (5.29) for a large enough number of coefficients $F = V_\alpha$, where this process continues until a certain stopping criterion is met (see [12] for some discussion regarding possible stopping criteria). Recall that the discretization of (5.29) leads to the matrix LCP given by (5.30), the solution of which is a numerical approximation to the original problem. We use the PSOR method described in 5.3.4 to obtain a solution of this matrix problem.

Now, for the discount factors given by (3.3), (3.4), and (3.5), we have to rely on an external analysis about the different aspects involved in the definitions of these quantities, typically a statistical analysis of the economic environment is sufficient. We therefore can use the results published by the different institutions mentioned in [1].

We mentioned earlier, in 5.1.1, that most of the players of the housing market are interested in just a few statistical moments, especially the expectation $E[V]$ and the variance $\text{Var}[V]$. For this reason, our main interest is to derive the expected value of the value of the mortgage, which we recall is the same as the first coefficient of the chaos expansion series (5.1).

The first set of numerical experiments are performed with a reasonable simplification assumption that will speed up the convergence of the algorithms involved. We will first assume that the short interest rate r is piecewise constant.

5.4.1 Numerical Results when r is Piecewise Constant

The short interest rate r is assumed to be piecewise constant, the break points defined at precisely the payment dates τ_k (remember that for the simplicity of the numerical simulations we work with the reverse time τ given by (2.33)), the valuation PDE (2.60) simplifies, as all the partial derivatives with respect to the variable r drop. We then solve the resulting matrix LCP, a simplification of (5.30), to obtain an approximation of $V_0(h, \tau)$.

Now, for the necessary model parameters, including those that appear in equations (2.1) and (2.2), we will use a set of values close to those used by the mortgage valuation literature such as [17], [15], and [25]. Our choices for this simplified version of the model are shown in

L	N_m	c	σ_h	δ	LTV
\$80,000	360	0.05	0.1	0.085	0.8

Table 5.1 Parameters used in the simplified model

table 5.1². Notice that we have a mortgage contract with maturity of $N_m = 360$ months, or 30 years, and fixed interest rate of $c = 0.05$, or 5 %.

In this section, we use $I = 200$ points in the h -axis and $N = 20$ in the t -axis for the discretization. We also set the minimum and maximum values for the value of the house price h as $h_{min} = 0$ and $h_{max} = 130^3$.

The first set of numerical results are given for the month when $k = 180$, that is to say, the month in between payment dates τ_{180} and τ_{181} . We also fix the time to be $\tau = 14.9417$ and the house price to be $h = 96.85$. Graphs of the numerical approximation of $V_0(h, \tau)$ are shown in figure 5.4.

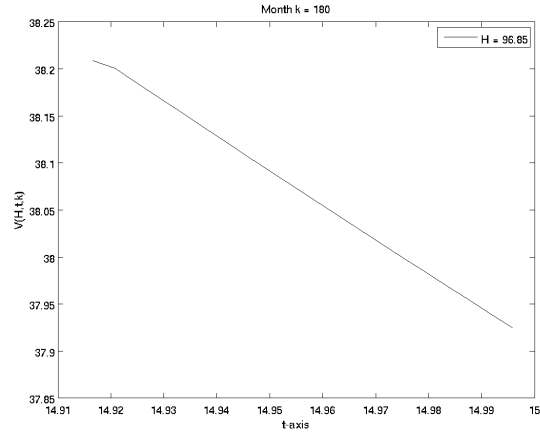
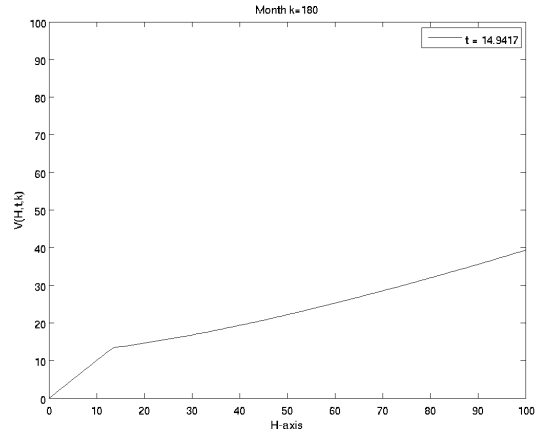
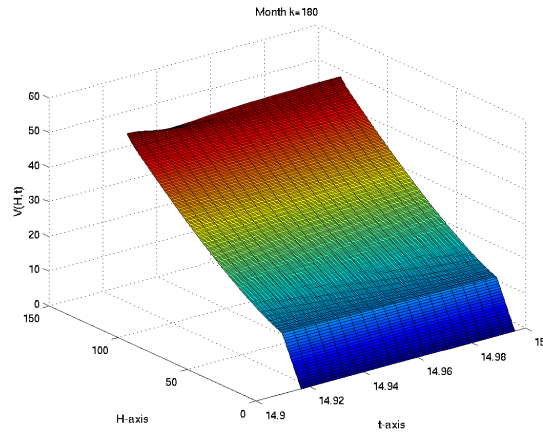
Since we are simulating the expected value of 30 year mortgage, we then compare the results presented in 5.4 to the numerical results we obtain for the simulation of another month, in this case for the last month when $k = 360$, and we fix the time to be $\tau = 29.94$ and the house price to be $h = 96.85$. We remark that the last month in the reverse τ time corresponds to the first month after the origination of the mortgage. Graphs of the numerical approximation of $V_0(h, \tau)$ for this month are shown in figure 5.5.

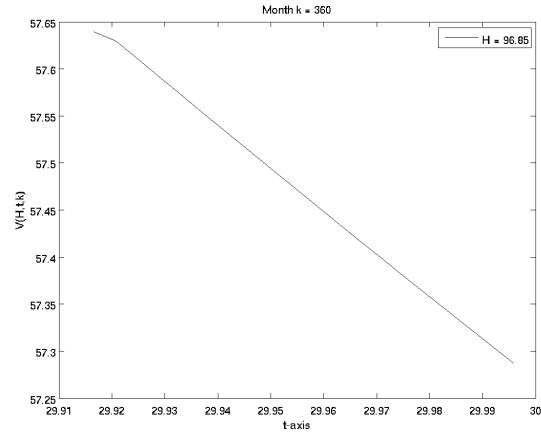
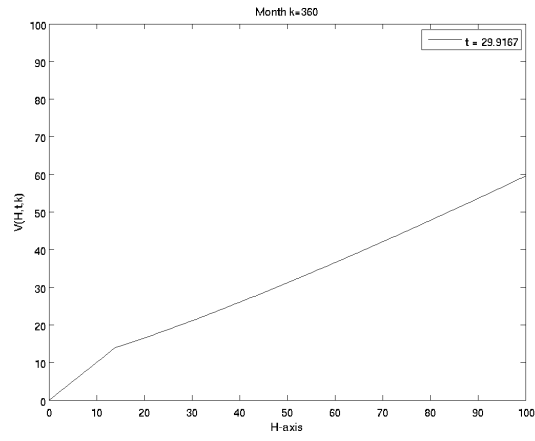
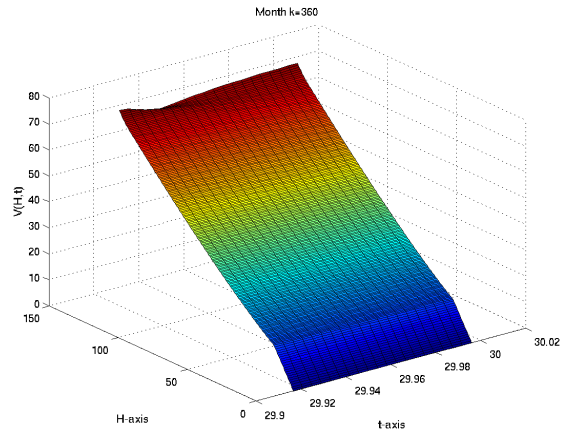
We now compare the graphs of the expected value of the mortgage $V_0(h, \tau)$ for the months $k = 180$ and $k = 360$, given in figures 5.4(c) and 5.5(c) respectively, and observe that the general shape of this surface is consistent with the graphs of the value of standard American options like those that we can find in [30]. This is consistent with the fact that, due to the free boundary condition (5.10), the function $V_0(h, \tau)$ has in essence the same properties as the value of an American put option.

In addition to this, notice as well that the values that V_0 takes are higher for $k = 360$ than for $k = 180$. This is indeed expected, since in the first case we presenting the graph

²LTV denotes the *loan-to-value ratio*, defined as $\frac{L}{h(0)}$

³The values of h and V_0 presented in this and the next section are all in thousands of dollars

(a) $V_0(h, \tau)$ vs τ (b) $V_0(h, \tau)$ vs h (c) $V_0(h, \tau)$ Figure 5.4 Numerical results for month $k = 180$

(a) $V_0(h, \tau)$ vs τ (b) $V_0(h, \tau)$ vs h (c) $V_0(h, \tau)$ Figure 5.5 Numerical results for month $k = 360$

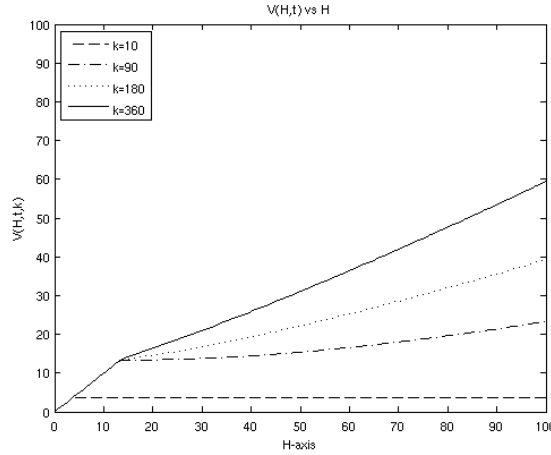


Figure 5.6 The effect of k on V_0

of $V_0(h, \tau)$ when τ is constrained to the interval $[\tau_{360}, \tau_{361})$, and in this time interval we are closer to the origination of the mortgage, which occurs precisely when $\tau = \tau_{361}$. The closer to origination, the higher the mortgage value must be, since a borrower has not yet reduced its debt good enough by making sufficient monthly payments. This property is more evident when we explore the effect that the change of the month in between payment dates k has on the expected value V_0 by running numerical simulations when k is 10, 90, 180, and 360 months. We fix a typical time τ and present the numerical results in figure 5.6.

Notice that the lowest graph in 5.6 corresponds to $k = 10$, while the highest corresponds to $k = 360$, and all the other graphs lie in between. This is a expected property of any mortgage value, and hence of its average value V_0 .

Next, we move on to compare different numerical simulations to see what effect the change in the mortgage interest rate c has on the average value V_0 . We fix the month to be $k = 180$, as well as all the parameters from table 5.1, and let c take on the values 0.045, 0.05, 0.055, and 0.09, or equivalently 4.5%, 5%, 5.5%, and 9%. We present the numerical results in figure 5.7

As it is expected, for higher mortgage interest rates c we obtain higher average mortgage values V_0 . This is natural, since a borrower with a higher interest rate has to potentially make higher monthly payments, making the mortgage more valuable.

Finally, to conclude the experimentation, we turn and look at the effect on the value of

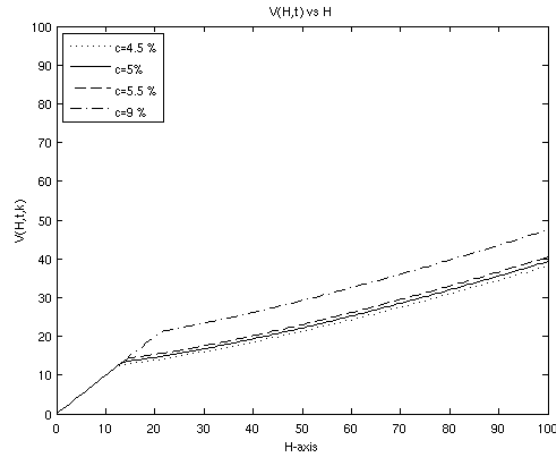


Figure 5.7 The effect of c on V_0 , when $k = 180$

the mortgage $V_0(h, \tau)$ for different loan amounts L , another parameter that is commonly used to compare different mortgage values. We fix the rest of the parameters from table 5.1 and also make $k = 180$ again. The loan value L is allowed to take on the values \$75,000, \$80,000, \$85,000, and \$90,000. The numerical results of this experiment are presented in figure 5.8.

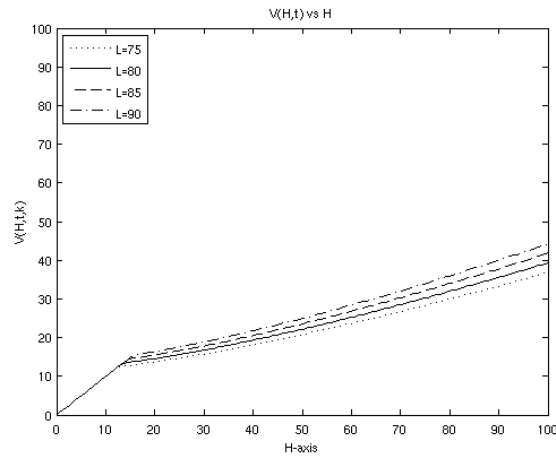


Figure 5.8 The effect of L on V_0 , when $k = 180$ months

Once again, the result is expected, as the lower graph in figure 5.8 corresponds to lower loan amount, while the higher graphs correspond to higher loan amounts.

The results presented in this subsection, for the very special case when r is piecewise constant, demonstrate that this thesis model produces average mortgage values that agree

L	Nm	c	σ_h	δ	σ_r	κ	θ	ρ	LTV
\$80,000	360	0.05	0.1	0.085	0.07	0.25	0.1	0	0.8

Table 5.2 Parameters used in the full model

with the general results of the previous mortgage valuation literature. This is indeed very promising and thus, to complete the supporting evidence of our model, we explore in the next subsection the situation when we do not have any simplification assumption on r .

5.4.2 Numerical Results for the Full Model

Similarly as in the previous subsection, we will choose similar parameters for the numerical simulations as those used in the previous mortgage valuation literature. Our choices for this parameters are shown in table 5.2.

Also, similarly as before, for the rest of the chapter we use $I = 200$ points in the h -axis, $J = 200$ in the r -axis, and $N = 20$ in the τ -axis for the discretization. Moreover, we also set the minimum and maximum values for the house price h to be again $h_{min} = 0$ and $h_{max} = 130$, while the minimum and maximum values for the short interest rate r are set to be $r_{min} = 0$ and $r_{max} = 30$.

First we simulate the average value of the mortgage $V_0(h, r, \tau)$ for different loan amounts L , with the rest of the parameters from 5.2 fixed. We once again let L be equal to \$75,000, \$80,000, \$85,000, and \$90,000. Moreover, we let $k = 180$ once more and also set the house price at $h = 96.85$. The numerical results obtained with these values are shown in figure 5.9.

Just as expected, the average mortgage value V_0 drops for lower loan amounts. Notice that the graphs presented in figure 5.9 are a different from those of the previous subsection, as we are now graphing the function V_0 against the reverse time τ .

Next, with the same setting as above, we present the numerical results that show the effect that different loan amounts L have on the average mortgage value $V_0(h, r, \tau)$. The results of this new numerical experiment are presented in figure 5.10.

Once again, the results are as expected, as the lower graph corresponds to a lower loan amount, while the higher graphs correspond to higher values of L . Moreover, the graphs

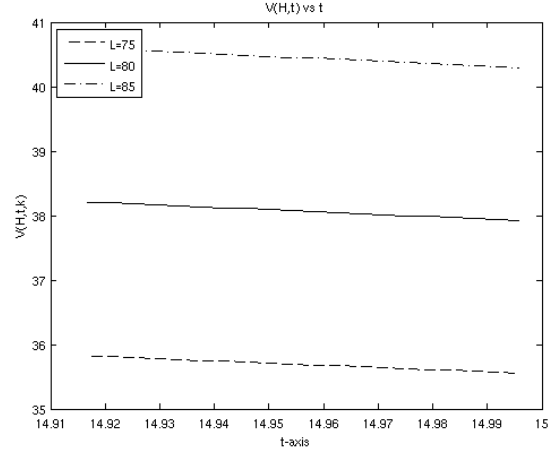


Figure 5.9 The effect of L on V_0 , when $k = 180$ and $h = 96.85$

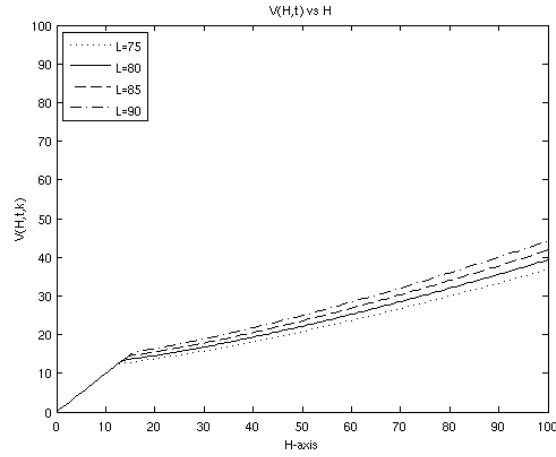


Figure 5.10 The effect of L on V_0 , when $k = 180$ and $\tau = 29.94$

present essentially the same behavior as those presented in figure 5.8.

Other numerical simulations produce similar results as those shown in figures 5.10 and 5.10 and we will omit presenting any more numerical results for the simplicity of the presentation. It suffices to say that the results support the practicability of the mortgage valuation model introduced in this thesis, as these results show desired properties that any good approximation to the average mortgage value should have. For instance, in all of the graphs shown in figures 5.4(b), 5.5(b), 5.6, 5.7, 5.8, and 5.10, the average mortgage value V_0 drops to zero after a certain value of the house price h . This behavior is consistent with the default property that a mortgage contract has, since for low values of h the expected mortgage value V_0 must also be low and eventually drop to 0 (when $h = 0$).

Therefore, to conclude this section, we remark that our mortgage valuation model produces reasonable and consistent results as those that are expected of all models, moreover, the results are different from the previous literature since we can now solve the more general problem of valuing mortgage contract when stochastic boundary conditions are considered. In this sense, the numerical evidence presented here supports the statement that, at the very least, the model of this thesis is an extension and an improvement over the option-theoretic models presented in the literature.

See appendix B for a pseudocode of the computer script used to run the simulations of this section.

CHAPTER 6. CONCLUDING REMARKS

In this chapter we make some remarks regarding the valuation of MBS.

6.1 Mortgage Backed Securities

This thesis introduced a new model and established a new framework for the valuation of mortgages in chapters 3 and 5. This contribution can be now taken into the valuation of some of the other financial contracts related to the housing market, in particular it extends easily to the valuation of *mortgage-backed securities (MBS)*. We present in this final chapter some remarks regarding the extension of our model to this, in general, harder problem.

A mortgage-backed security, or as it is commonly referred to as MBS, is a financial contract that represents a claim on the cash flows originating from a group of mortgage loans, which are most commonly backed by residential property.

The process to create such a contract begins first when mortgage loans are purchased from banks, mortgage companies, and other mortgage contract originators. Then, these loans are assembled into groups called *pools*. A mortgage-backed security then represent a claim on the principal balance and monthly payments of the loans belonging to the pool, through a process called *securitization*. These securities are then typically as if they are bonds, although in the years leading to the housing market crisis of 2007, the financial markets created a broad variety of different securities that derive their value from mortgage pools.

Securitization can be described as a financial process that distributes risk among the different mortgages in a pool of mortgages. That is to say, from the perspective of an investor, buying a claim on a single mortgage is riskier than buying a claim on a pool of them. Indeed, with a single mortgage backing his claim, an investor would lose the stream of payments if the

underlying borrower either defaults or prepays, therefore he is exposed to high risk. On the other hand, with a pool of mortgages backing his claim, the investor has less risk, as it is unlikely that all the underlying borrowers would either default or prepay simultaneously, therefore he has less risk.

The assumption about the unlikeliness of simultaneous default or prepayment is only true if the underlying mortgages in the corresponding pool are all very similar and of rather high credit quality. The crisis of 2007 showed that when the mortgages in a pool are of poor quality, for instance *sub-prime mortgages*, then the event of near simultaneous default is very likely.

Now, the qualifier *pass-through MBS* is typically used to distinguish the basic kind of MBS from other type of mortgage backed contracts, this is analogous to the *vanilla option* used to differentiate the basic type of options from the more sophisticated ones. So, a pass-through security typically has the same type of underlying mortgage loan and this mortgages are similar enough with respect to their maturity and contract interest rate to allow cash flows to be projected as if the pool were a single mortgage. This is precisely the kind of MBS that our valuation model naturally extends to.

For more on pass-through MBS and other type of mortgage-backed securities see [11] and [13].

6.2 Valuation of Pass-through MBS

We conclude this chapter with a brief outline of how the mortgage valuation of this thesis can be used for the valuation of pass-through MBS.

First we remark that an estimate of the cash flows coming from a pool of mortgages can be computed by means of the *weighted average maturity (WAM)* and the *weighted average coupon (WAC)*. To define these two quantities, first consider a pool of mortgage contracts consisting of m loans, denoted by L_1, L_2, \dots, L_m . Next, let PB_1, PB_2, \dots, PB_m denote the principal balances of these contracts respectively, at the time of issuance of the MBS (when $t = 0$). We define the *weight* of each loan in the pool as

$$w_i = \frac{PB_i}{\sum_{j=1}^m PB_j}, \quad 1 \leq i \leq m.$$

Then, if we let T_1, T_2, \dots, T_m denote the corresponding maturities of the loans in the pool, the weighted average maturity is defined by:

$$\text{WAM} = \sum_{j=1}^m w_j T_j. \quad (6.1)$$

Next, if we let c_1, c_2, \dots, c_m denote the corresponding coupons of the mortgages in the pool, the weighted average coupon in turn is defined as:

$$\text{WAC} = \sum_{j=1}^m w_j c_j. \quad (6.2)$$

Notice that equations (6.1) and (6.2) are just two weighted averages, which are used to characterize and describe a pass-through MBS. This characterization consists in visualizing the pass-through MBS as a single mortgage with maturity given by (6.1) and fixed interest rate given by (6.2).

Given this characterization, we can use the approach of this thesis to find the value of a pass-through MBS as if it is a single mortgage. An alternative and also natural approach is to find the value of each of the mortgages in the underlying pool by using the WAM, given in (6.1), and the WAC, given in (6.2), as the maturity and interest rate respectively of each these mortgages. The a pass-through MBS $V_{MBS}(h, r, t)$ can then be found as

$$V(h, r, t) = \sum_{j=1}^m w_j V_j(h, r, t), \quad (6.3)$$

where $V_j(h, r, t)$ is the value of the mortgages contract with loan L_j , for $1 \leq j \leq m$.

The approach given by (6.3) allows us to rate each of the mortgages with value V_j with different discount factors λ_j , all given by formula (3.2), therefore we can consider a pool of mortgages with different credit quality.

In conclusion, the mortgage valuation model presented in this thesis provides a much better and up-to-date value for each of the individual mortgages in a pool underlying a pass-through MBS. Unlike previous models that did not properly account for current conditions of the economy, our model incorporates conditions like those of the economic crisis of 2007, when the credit crunch in the US economy was greatly due to the overpricing of MBS, especially those based on subprime mortgages.

APPENDIX A. MONTHLY PAYMENT AND PRINCIPAL BALANCE

We present a short derivation of the formulas for the monthly payment MP , given by (2.3), and the principal balance $PB(i)$, given by (2.4). Both MP and $PB(i)$ are determined by the following financial condition: The sum of the present value of all the future monthly payments, which are discounted by the mortgage interest rate, must be equal to the original principal balance. So, writing this statement in mathematical terms give:

$$\begin{aligned} L &= \frac{MP}{(1 + \frac{c}{12})} + \frac{MP}{(1 + \frac{c}{12})^2} + \cdots + \frac{MP}{(1 + \frac{c}{12})^{N_m}} = \sum_{i=1}^{N_m} \frac{MP}{(1 + \frac{c}{12})^i} \\ &= \frac{MP}{(1 + \frac{c}{12})} \left[\frac{1 - \frac{1}{(1 + \frac{c}{12})^{N_m}}}{1 - \frac{1}{(1 + \frac{c}{12})}} \right] = \frac{MP}{(1 + \frac{c}{12})^{N_m}} \left[\frac{(1 + \frac{c}{12})^{N_m} - 1}{\frac{c}{12}} \right]. \end{aligned}$$

Hence, the monthly payment MP is found to satisfy

$$MP = \frac{L (1 + \frac{c}{12})^{N_m} (\frac{c}{12})}{(1 + \frac{c}{12})^{N_m} - 1}. \quad (\text{A.1})$$

As for the unpaid principal $PB(i)$, we can rephrase the financial conditions as: The amount owed on the loan at the end of every month is equal to the amount owed on the previous month, plus the interest on this amount, minus the fixed amount paid every month. Therefore, after the i -th payment date, $PB(i)$ is given by the expression:

$$PB(i) = L \left[\frac{1 - (1 + \frac{c}{12})^{-(N_m - i)}}{1 - (1 + \frac{c}{12})^{-N_m}} \right] = \frac{L \left[(1 + \frac{c}{12})^{N_m} - (1 + \frac{c}{12})^i \right]}{(1 + \frac{c}{12})^{N_m} - 1}. \quad (\text{A.2})$$

APPENDIX B. PSEUDOCODES

We present the main sections of the computer code written to perform the numerical simulations in section 5.4.2. The first pseudocode is part of the main computer script, and then second one is the external PSOR method solver. We use a simplification of MATLAB syntax in both of them.

- Main pseudocode:

```

for i=2:I+1; j=2:J
  Define Crank-Nicolson coefficients;
end; end

for j=3:J; for j=2:J; for j=2:J-1
  Define bands of coefficient matrix;
end

for j=1:J+1
  V(:,j,1) = min(MP,H);
end

%Main loop:
for k=1:Nm; for nn=1:N-1;
  n = nn + N*(k-1);
  V(2:I+1,1,n) = min(H(2:I+1),TD(nn,k));

for i=2:I; for j=2:J
  Define the right-hand-side vector RHS;
end; end

tolerance = 1e-12;

```

```

F0=min(V(2:I+1,2:J,n),TD(nn+1,k));
[V(2:I+1,2:J,n+1),loops] = ...
PSOR(A(:, :, 3:J),B(:, :, 2:J),C(:, :, 2:J-1),G(:, :, n),TD(nn+1,k),F0,tol);
for j=1:J+1
V(:,j,n+2) = min(V(:,j,n+1) + MP, H');
end; end

```

- PSOR solver:

```

while(err>tol)
    Y(:,1) = BB(:, :, 1)\(G(:,1)-CC(:, :, 1)*F(:,2));
    Y(:,1) = min(const,(Y(:,1)-F(:,1))*omega + F(:,1));
    for j=2:JJ-1
        Y(:,j) = BB(:, :, j)\(G(:,j)-AA(:, :, j-1)*Y(:,j-1)-CC(:, :, j)*F(:,j+1));
        Y(:,j) = min(const,(Y(:,j)-F(:,j))*omega + F(:,j));
    end
    Y(:,JJ) = BB(:, :, JJ)\(G(:,JJ)-AA(:, :, JJ-1)*Y(:,JJ-1));
    Y(:,JJ) = min(const,(Y(:,JJ)-F(:,JJ))*omega + F(:,JJ));
    err = norm(Y-F);
    loops = loops + 1;
    F=Y;
end

```

BIBLIOGRAPHY

- [1] Bernard Baumohl. *The secrets of economic indicators: hidden clues to future economic trends and investment opportunities*. Wharton School Publishing, second edition, 2007.
- [2] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637, 1973.
- [3] M. J. Brennan and E. S. Schwartz. Determinants of GNMA Mortgage Prices. *Journal of AREUEA*, 13(3):209–228, 1985.
- [4] Damiano Brigo and Fabio Mercurio. *Interest Rate Models-Theory and Practice with Smile, Inflation and Credit*. Springer, Berlin, 2nd edition, 2006.
- [5] Andrew J. G. Cairns. *Interest Rate Models: An Introduction*. Princeton University Press, Princeton, 2004.
- [6] R. Chen and T. Yang. The relevance of interest rate processes in pricing mortgage-backed securities. *Journal of Housing Research*, 6(2):315–322, 1995.
- [7] John C. Cox, Ingersoll Jonathan E., and Ross Stephen A. A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407, 1985.
- [8] John Crank. *Free and moving boundary problems*. Oxford University Press, New York, 1984.
- [9] L. Uri Dothan. On the term structure of interest rates. *Journal of Financial Economics*, 6(1):59 – 69, 1978.

- [10] C. M. Elliott and J. R. Ockendon. *Weak and Variational Methods for Free and Moving Boundary Problems*. Pitman, 1982.
- [11] Frank J. Fabozzi. *The Handbook of Mortgage-Backed Securities*. McGraw-Hill, 5th edition, 2001.
- [12] Helge Holden, Bernt Øksendal, Jan Ubøe, and Tusheng Zhang. *Stochastic partial differential equations: A modeling, white noise functional approach*. Probability and its Applications. Birkhäuser Boston Inc., Boston, 1996.
- [13] Joseph Hu. *Basics of Mortgage-Backed Securities*. Frank J. Fabozzi Associates, Pennsylvania, second edition, 2001.
- [14] J. B. Kau, D.C. Keenan, W.J. Muller III, and J.F. Epperson. Pricing commercial mortgages and their mortgage-backed securities. *The Journal of Real Estate Finance and Economics*, 3(4):333–56, December 1990.
- [15] J. B. Kau, D.C. Keenan, W.J. Muller III, and J.F. Epperson. The valuation at origination of fixed-rate mortgages with default and prepayment. *The Journal of Real Estate Finance and Economics*, 11(1):5–36, July 1995.
- [16] James B. Kau and Donald C. Keenan. An overview of the option-theoretic pricing of mortgages. *Journal of Housing Research*, 6(2):217–44, 1995.
- [17] James B. Kau, Donald C. Keenan, III Muller, Walter J., and James F. Epperson. A generalized valuation model for fixed-rate residential mortgages. *Journal of Money, Credit and Banking*, 24(3):279–299, 1992.
- [18] C. T. Kelley. *Iterative Methods for Linear and Nonlinear Equations*. Society for Industrial and Applied Mathematics, Philadelphia, 1995.
- [19] David Kinderlehrer and Guido Stampacchia. *An Introduction to Variational Inequalities and Their Applications*, volume 31 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics, Philadelphia, 2000. Reprint of the 1980 original.

- [20] M. D. Koulisianis and T. S. Papatheodorou. Improving projected successive overrelaxation method for linear complementarity problems. *Appl. Numer. Math.*, 45(1):29–40, 2003.
- [21] M. D. Koulisianis and T. S. Papatheodorou. Pricing of American options using linear complementarity formulation: methods and their evaluation. *Neural, Parallel Sci. Comput.*, 11(4):423–444, 2003.
- [22] Peter Newman, Murray Milgate, and John Eatwell, editors. *The New Palgrave Dictionary of Money and Finance*. Stockton Press, New York, 1992.
- [23] Bernt Øksendal. An introduction to malliavin calculus with applications to economics. Unpublished lecture notes, available at <http://hdl.handle.net/2330/841>, 1997.
- [24] Bernt Øksendal. *Stochastic Differential Equations: An Introduction with Applications*. Springer-Verlag Heidelberg, New York, 5th edition, 2002.
- [25] N. J. Sharp. *Advances in Mortgage Valuation: An Option-Theoretic Approach*. PhD thesis, School of Mathematics, University of Manchester, 2006.
- [26] Toru Sugimura. Valuation of residential mortgage-backed securities with default risk using an intensity-based approach. *Asia-Pacific Financial Markets*, 11(2):185–214, 2004.
- [27] J. W. Thomas. *Numerical Partial Differential Equations: Finite Difference Methods*. Springer-Verlag, New York, 1995.
- [28] Oldrich Vasicek. An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188, 1977.
- [29] Paul Wilmott, Jeff Dewynne, and Sam Howison. *Option Pricing: Mathematical Models and Computation*. Oxford Financial Press, 1993.
- [30] Paul Wilmott, Sam Howison, and Jeff Dewynne. *The Mathematics of Financial Derivatives, A Student Introduction*. Cambridge University Press, 1995.

ACKNOWLEDGEMENTS

I wish to express my gratitude to professor Steven Hou, my academic adviser, for all the patience and support while I was his student at Iowa State University. Steve has taught me so many valuable things, not only about mathematics, but also about life. I am very fortunate to have met him.

I would also like to thank my thesis defense committee, professors Scott Hansen, Hailiang Liu, Sung-Yell Song, and Ananda Weerasinghe, for their valuable advices and time.

There have been many people that have helped me all these years, I extend my appreciation to all of them.

Finally, I thank my family and my dearest friend Ahyoung for all their love.